Serie 7: Bosones y Cuasipartículas

1.

$$N = \sum_j \langle n_j \rangle = \sum_\varepsilon \frac{1}{z^{-1}e^{\beta\varepsilon}-1}$$ para estadísticas de Bose-Einstein

$$\Rightarrow \quad \sum_p \rightarrow \frac{A}{h^2}\int d^2p \rightarrow \frac{A}{h^2}\int\int_{-\infty}^{+\infty} dp_x\, dp_y = \frac{A}{h^2}\int_0^\infty\int_0^{2\pi} p.d\theta.dp$$

$$\frac{2\pi A}{h^2}\int_0^\infty p.dp \qquad p^2 = 2m\varepsilon$$

$$\frac{2\pi m A}{h^2}\int_0^\infty d\varepsilon \langle n_\varepsilon\rangle \qquad 2p\,dp = 2m\,d\varepsilon \qquad p.dp = m\,d\varepsilon$$

$$N = \frac{2\pi m A}{h^2}\int_0^\infty \frac{d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1}$$

$\beta\varepsilon = x$, $\beta d\varepsilon = dx$

$$N = \frac{2\pi m A}{h^2}\frac{1}{\beta}\int_0^\infty \frac{dx}{z^{-1}e^{x}-1}$$

$$N = \frac{A}{\lambda^2}\int_0^\infty \frac{dx}{z^{-1}e^{x}-1} = \frac{A}{\lambda^2}\int_0 \frac{dx}{z^{-1}e^{x}-1} + \underbrace{\frac{1}{z^{-1}-1}}_{\frac{z}{1-z} = N_0}$$

, pero como:

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^{x}-1} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \cdots$$

$$\Rightarrow \quad g_{(1)}(z) = \frac{1}{\underbrace{\Gamma(1)}_{=1}}\int_0^\infty \frac{dx}{z^{-1}e^{x}-1} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$\Rightarrow \quad \frac{N-N_0}{A} = \frac{1}{\lambda^2}\cdot\sum_{n=1}^{\infty}\frac{z^n}{n}$$

Luego, como $\sum_n^\infty \left(\frac{z^n}{n}\right)$ no está acotada para $z=1$ se sigue que no hay cota para el # de partículas en los estados excitados. Entonces puedo seguir agregando partículas al sistema sin que se "llenen totalmente" los niveles excitados y se produzca una condensación que pueble macroscópicamente el fundamental.

Nótese que, pese a no ser necesaria la extracción del término $N_0 = \frac{z}{1-z}$ de la integral, por no hallarse ponderado ese nivel n_0 con cero, puede realizarse porque es una contribución de medida nula a la misma y entonces tenemos

$$\sum_p \longrightarrow \frac{\rho.A}{\lambda^2}\int d\varepsilon = \frac{A}{\lambda^2}\int dx$$

$$\frac{\lambda^2}{A}\sum_p \longleftarrow \int dx$$

$$\frac{\lambda^2}{A}N_0 \longleftarrow \underbrace{\int_0^0 dx\langle n_0\rangle}$$ $$\Rightarrow \quad \frac{1}{z^{-1}-1} \equiv N_0$$

Esto es una notación, tomar con pinzas

$W \neq N_E \cdot N_0$

2.

gas de Bose-Einstein con partículas con g.L. internos

$\Rightarrow$ 2 g.L. internos $\longrightarrow \begin{array}{l} \epsilon_1 \neq 0 \\ \epsilon_0 = 0 \end{array} \Rightarrow$

hamiltoniano para una partícula $\nearrow$ $\varepsilon_i = \dfrac{p_i^2}{2m} + \epsilon_i$ $\qquad$ $i=1 \rightarrow \epsilon_1$ una constante; $i=0 \rightarrow \epsilon_0 = 0$

$$\langle n_j \rangle = \frac{1}{z^{-1} e^{\beta \varepsilon} - 1}$$

Para $N > N_E^{MAX}$ habrá condensación de Bose $\Rightarrow$ En el caso 3-D con $\varepsilon = p^2/2m$ se tenía:

$$N > N_E^{MAX} = V \frac{(2\pi m k T)^{3/2}}{h^3} g_{3/2}(1)$$

$$\frac{N}{V \cdot g_{3/2}(1)} \frac{h^3}{(2\pi m k)^{3/2}} > T^{3/2}$$

$$T_c^0 \equiv \left(\frac{N}{V g_{3/2}(1)}\right)^{2/3} \frac{h^2}{2\pi m k} > T$$

Necesito $T < T_c^0$ para que comienze a ocurrir la condensación de Bose, donde T_c^0 es la temperatura crítica de un gas sin g.L. internos. En nuestro caso será:

$$N = \sum_j \langle n_j \rangle = \sum_\epsilon \langle n_\varepsilon \rangle = \sum_\varepsilon \frac{1}{z^{-1} e^{\beta \varepsilon} - 1}$$

$$= \sum_{p,i} \frac{1}{z^{-1} e^{\beta p^2/2m + \beta \epsilon_i} - 1}$$

$$N = \sum_p \left(\underset{Ⓐ}{\frac{1}{z^{-1} e^{\beta p^2/2m + \beta \epsilon_1} - 1}} + \underset{Ⓑ}{\frac{1}{z^{-1} e^{\beta p^2/2m} - 1}} \right)$$

El paso al ~~continuo~~ es:

Ⓐ $\displaystyle \sum_p \rightarrow \frac{V}{h^3} \int d^3p = \iiint \frac{V}{h^3} p^2 \operatorname{sen}\theta \, d\theta \, d\varphi \, dp = \frac{4\pi V}{h^3} \int_0^\infty p^2 dp$

$\dfrac{p^2}{2m} + \epsilon_1 = \varepsilon$ $\qquad$ $\displaystyle = \frac{4\pi V}{h^3} \int_{\epsilon_1}^\infty (2m)^{1/2} (\varepsilon - \epsilon_1)^{1/2} d\varepsilon \, m$

Ⓑ $p = \sqrt{(\varepsilon - \epsilon_1) 2m}$; $p \cdot dp = m \cdot d\varepsilon$ $\qquad$ $\displaystyle \frac{4\pi V}{h^3} 2^{1/2} m^{3/2} \int_0^\infty \varepsilon^{1/2} \qquad = \frac{4\pi V}{h^3} 2^{1/2} m^{3/2} \int_{\epsilon_1}^\infty (\varepsilon - \epsilon_1)^{1/2} d\varepsilon$

$\dfrac{p^2}{2m} = \varepsilon$ $\qquad$ $\dfrac{p \, dp}{m} = d\varepsilon \rightarrow \displaystyle \frac{4\pi V}{h^3} \int_0^\infty (2m\varepsilon)^{1/2} m \, d\varepsilon$

$$N = \frac{4\pi V \sqrt{2} \, m^{3/2}}{h^3} \left[\int_{\epsilon_1}^\infty \frac{(\varepsilon - \epsilon_1)^{1/2} d\varepsilon}{z^{-1} e^{\beta \varepsilon} - 1} + \frac{(\varepsilon)^{1/2} d\varepsilon}{z^{-1} e^{\beta \varepsilon} - 1} \right]$$

$$N = \frac{4\pi V \sqrt{2} \, m^{3/2}}{h^3} \left[\int_0^\infty \underbrace{\frac{\varepsilon'^{1/2} \cdot d\varepsilon'}{z^{-1} e^{\beta \varepsilon'} \cdot e^{\beta \epsilon_1} - 1}}_{} + \int_0^\infty \frac{\varepsilon^{1/2} \cdot d\varepsilon}{z^{-1} e^{\beta \varepsilon} - 1} \right] \qquad \varepsilon = x/\beta$$

$\varepsilon - \epsilon_1 = \varepsilon'$; $d\varepsilon = d\varepsilon'$ $\qquad$ $z' \equiv (z e^{-\beta \epsilon_1})^{-1} e^{\beta \varepsilon'} - 1 + \displaystyle \frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2} dx}{z^{-1} e^x - 1}$ $\qquad$ $\beta \varepsilon = x$; $d\varepsilon = \dfrac{dx}{\beta}$

$$\varepsilon - \epsilon_1 = \varepsilon' \qquad \Gamma(1+1/2) = 1/2 \cdot \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$N = \frac{4\pi V\, 2^{1/2} m^{3/2}}{h^3}\left[\int_0^\infty \frac{\varepsilon'^{1/2}\, d\varepsilon'}{z'^{-1} \cdot e^{\beta\varepsilon'} - 1} + \frac{1}{\beta^{3/2}}\, \Gamma(3/2) \cdot g_{3/2}(z) \right]$$

$$\frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}\, dx}{z'^{-1} \cdot e^{x} - 1} + \frac{\sqrt{\pi}}{2\beta^{3/2}} \cdot g_{3/2}(z)$$

$$\frac{1}{\beta^{3/2}} \frac{\sqrt{\pi}}{2} \cdot g_{3/2}(z e^{-\beta\epsilon_1}) + \frac{1}{\beta^{3/2}} \frac{\sqrt{\pi}}{2}\, g_{3/2}(z)$$

$$g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots$$

$$g_{3/2}(z') = z \cdot e^{-\beta\epsilon_1} + \frac{z^2 \cdot e^{-2\beta\epsilon_1}}{2^{3/2}} + z^3 \cdot \frac{e^{-3\beta\epsilon_1}}{3^{3/2}} + \dots$$

si $z = 1 \rightarrow \quad g_{3/2}(z') = g_{3/2}(e^{-\beta\epsilon_1}) = e^{-\beta\epsilon_1} + \frac{e^{-2\beta\epsilon_1}}{2^{3/2}} + \dots$

Sea ahora $z = 1 \Rightarrow$

$$N_\varepsilon^{MAX} = N - N_0 = \frac{V}{h^3}(2\pi m)^{3/2} (kT)^{3/2} \left[g_{3/2}(e^{-\beta\epsilon_1}) + g_{3/2}(1) \right]$$

$\Rightarrow$ con $N > N_\varepsilon^{MAX}$ será:

$$N > V\frac{(2\pi m)^{3/2}}{h^3} (kT)^{3/2} g_{3/2}(1) \left[1 + \frac{g_{3/2}(e^{-\beta\epsilon_1})}{g_{3/2}(1)} \right]$$

$$\frac{N}{V \cdot g_{3/2}(1)} \cdot \frac{h^3}{(2\pi m k)^{3/2}} \cdot \frac{1}{\left(1 + \frac{1}{g_{3/2}(1)} [g_{3/2}(e^{-\beta\epsilon_1})]\right)} > T^{3/2}$$

$$T_c^{3/2} = \left(\frac{N}{V \cdot g_{3/2}(1)}\right) \frac{h^3}{(2\pi m k)^{3/2}} \cdot \frac{1}{\left(1 + \frac{1}{g_{3/2}(1)} \cdot [g_{3/2}(e^{-\beta\epsilon_1})]\right)}$$

$$\left(\frac{T_c^0}{T_c}\right)^{3/2} = \cancel{\frac{N}{V \cdot g_{3/2}(1)}}\, \cancel{\frac{h^3}{(2\pi m k)^{3/2}}} \cdot \cancel{\frac{V \cdot g_{3/2}(1)}{N}} \cdot \cancel{\frac{(2\pi m k)^{3/2}}{h^3}} \left[1 + \frac{1}{g_{3/2}(1)} [g_{3/2}(e^{-\beta\epsilon_1})] \right]$$

$$\boxed{\left(\frac{T_c^0}{T_c}\right)^{3/2} = 1 + \frac{1}{2{,}612} \cdot \left(e^{-\beta\epsilon_1} + \frac{e^{-2\beta\epsilon_1}}{2^{3/2}} + \dots \right)}$$

donde $\beta = \frac{1}{kT_c}$

3. gas de bosones spín $s=1$, temperatura T, densidad $1/v$

momento magnético $\mu = m_0 S_z$, campo H

$S_z = -1, 0, 1$

a) $$\varepsilon = \sum_i^N \frac{p_i^2}{2m} - m_0 S_{z_i} H$$

$$N = \sum_p \underbrace{\frac{1}{z^{-1} e^{\beta p^2/2m} - 1}}_{S_z=0} + \underbrace{\frac{1}{z^{-1} e^{\beta(p^2/2m - m_0 H)} - 1}}_{S_z=+1} + \underbrace{\frac{1}{z^{-1} e^{\beta(p^2/2m + m_0 H)} - 1}}_{S_z=-1}$$

$$N = \sum_p \frac{1}{z^{-1} e^{\beta p^2/2m} - 1} + \sum_p \frac{1}{z'^{-1} e^{\beta p^2/2m} - 1} + \sum_p \frac{1}{z^{-1} e^{\beta m_0 H} e^{\beta p^2/2m} - 1}$$

$$z^{-1} e^{-\beta m_0 H} = z^{-\beta(\mu + m_0 H)} \Rightarrow z' = e^{\beta(\mu + m_0 H)}$$

$$z^{-1} e^{\beta m_0 H} = z^{-\beta(\mu - m_0 H)} \qquad z' = z\, e^{\beta m_0 H}$$

$$e^{-\beta m_0 H} z' = z$$

$$N = \sum_p \underbrace{\frac{1}{z'^{-1} e^{\beta m_0 H} e^{\beta p^2/2m} - 1}}_{\langle n_p \rangle_{S_z=0}} + \sum_p \underbrace{\frac{1}{z'^{-1} e^{\beta p^2/2m} - 1}}_{\langle n_p \rangle_{S_z=+1}} + \sum_p \underbrace{\frac{1}{z'^{-1} e^{2\beta m_0 H} e^{\beta p^2/2m} - 1}}_{\langle n_p \rangle_{S_z=-1}}$$

b)

El pasaje al continuo será:

$$\sum_p \to \frac{V}{h^3}\int d\vec{p}^3 = \frac{V}{h^3} 4\pi \int p^2\, dp = \frac{4\pi V}{h^3} 2^{1/2} m^{3/2} \int_0^\infty \varepsilon^{1/2}\, d\varepsilon \Rightarrow$$

$$\frac{N}{V} = \frac{4\pi}{h^3} 2^{1/2} m^{3/2} \left[\int_0^\infty \frac{\varepsilon^{1/2}}{z'^{-1} e^{\beta m_0 H} e^{\beta\varepsilon} - 1}\, d\varepsilon + \int_0^\infty \frac{\varepsilon^{1/2}}{z'^{-1} e^{\beta\varepsilon} - 1}\, d\varepsilon + \int_0^\infty \frac{\varepsilon^{1/2}}{z'^{-1} e^{2\beta m_0 H} e^{\beta\varepsilon} - 1}\, d\varepsilon \right]$$

$$\beta\varepsilon = x \qquad d\varepsilon = \frac{1}{\beta} dx$$

$$\left[\frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}\, dx}{z'^{-1} e^{\beta m_0 H} e^{x} - 1} + \frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}}{z'^{-1} e^{x} - 1} dx + \frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}}{z'^{-1} e^{2\beta m_0 H} e^{x} - 1} dx \right]$$

$$\frac{N}{V} = \frac{\cancel{4}^{2}\pi}{h^3} 2^{1/2} m^{3/2} \frac{1}{\beta^{3/2}} \left[g_{3/2}(z' e^{-\beta m_0 H}) + g_{3/2}(z') + g_{3/2}(z' e^{-2\beta m_0 H}) \right] \frac{\sqrt{\pi}}{\cancel{2}}$$

$$\frac{N}{V} = \frac{(2\pi m k T)^{3/2}}{h^3} \left[g_{3/2}(z' e^{-\beta m_0 H}) + g_{3/2}(z') + g_{3/2}(z' e^{-2\beta m_0 H}) \right]$$

$$\frac{PV}{kT} = \ln Z_{GC} = \ln \left(\prod_{\varepsilon=0}^{\infty} \frac{1}{1 - e^{-\beta\varepsilon} z} \right) = -\sum_\varepsilon \ln(1 - e^{-\beta\varepsilon} z)$$

$s_z=+1$ $\qquad s_z=-1$

$$\frac{PV}{kT} = -\sum_p \left[\ln\left(1-e^{-\beta\frac{p^2}{2m}}\cdot z\right) + \ln\left(1-e^{-\beta\frac{p^2}{2m}+\beta m_oH}\cdot z\right) + \ln\left(1-e^{-\beta\frac{p^2}{2m}-\beta m_oH}\cdot z\right)\right]$$

defino $\quad z' = e^{\beta\mu+\beta m_oH} = z\,e^{\beta m_oH} = z\,e^{x}$

$$e^{-x}z' = z$$

$$= -\sum_p \ln\left(1-e^{-\beta\frac{p^2}{2m}}z'e^{-x}\right) + \ln\left(1-e^{-\beta\frac{p^2}{2m}}\cdot z'\right) + \ln\left(1-e^{-\beta\frac{p^2}{2m}}z'e^{-2x}\right)$$

↓ al continuar con relación de dispersión $p^2/2m = \varepsilon \rightarrow$

$$\frac{PV}{kT} = -\frac{4\pi V\, 2^{1/2} m^{3/2}}{h^3}\int_0^\infty \varepsilon^{1/2}d\varepsilon\left[\ln\left(1-e^{-\beta\varepsilon}z'e^{-x}\right) + \underbrace{\ln\left(1-e^{-\beta\varepsilon}\cdot z'\right)}_{\text{la integro por partes}} + \ln\left(1-e^{-\beta\varepsilon}z'e^{-2x}\right)\right]$$

$$\int_0^\infty \varepsilon^{1/2}d\varepsilon\cdot\ln\left(1-e^{-\beta\cdot\varepsilon}\cdot z'\right) = \ln\left(1-e^{-\beta\varepsilon}z'\right)\frac{\varepsilon^{3/2}}{3/2}\Bigg|_0^\infty - \int_0^\infty -\frac{\beta\varepsilon^{3/2}/_{3/2}}{z'^{-1}e^{\beta\varepsilon}-1}\cdot d\varepsilon$$

$u = \ln(1-e^{-\beta\varepsilon}\cdot z')$ $\qquad dv = \varepsilon^{1/2}\,d\varepsilon$

$du = \dfrac{1}{1-e^{-\beta\varepsilon}\cdot z'}\, z' e^{-\beta\varepsilon}\cdot\beta\, d\varepsilon$ $\qquad V = \varepsilon^{3/2}/3/2$

$$= +\frac{1}{3/2}\int_0^\infty \frac{\cancel{\beta}\, x^{3/2}\, dx}{z'^{-1}e^x - 1}\cdot\frac{1}{\cancel{\beta}}\frac{1}{\beta^{3/2}} \qquad \varepsilon^{3/2} = \frac{x^{3/2}}{\beta^{3/2}}$$

$$du = \frac{-\beta\cdot e^{-\beta\varepsilon}z'}{1-e^{-\beta\varepsilon}\cdot z'} = \frac{+z'^{-1}e^{\beta\varepsilon}(-\beta e^{-\beta\varepsilon})}{+z'^{-1}e^{\beta\varepsilon}-1} = -\frac{\beta}{z'^{-1}e^{\beta\varepsilon}-1}\cdot d\varepsilon$$

$\beta\varepsilon = x$

$\beta d\varepsilon = dx$

$\Gamma(5/2)$

$\Gamma(2+1/2)$

$= \frac{3\cdot 1}{4}\sqrt{\pi}$

$$= \frac{1}{\frac{3}{2}\beta^{3/2}}\Gamma(5/2)\, g_{5/2}(z')$$

$$\int_0^\infty \varepsilon^{1/2}d\varepsilon\cdot\ln\left(1-e^{-\beta\varepsilon}z'\right) = \frac{\cancel{2}^{1}}{\cancel{3}\beta^{3/2}}\frac{\cancel{3}}{\cancel{4}_{2}}\sqrt{\pi}\, g_{5/2}(z')$$

$$\frac{P}{kT} = \frac{\cancel{4}^{2}\pi\, 2^{1/2}m^{3/2}}{h^3}\frac{\pi^{1/2}}{\cancel{4}}\frac{1}{\beta^{3/2}}\left[g_{5/2}(z'e^{-x}) + g_{5/2}(z') + g_{5/2}(z'e^{-2x})\right]$$

$$\frac{P}{kT} = \frac{(2\pi m kT)^{3/2}}{h^3}\left[g_{5/2}(z'e^{-x}) + g_{5/2}(z') + g_{5/2}(z'e^{-2x})\right]$$

$$\boxed{\frac{P\cdot V}{NkT} = \frac{\left(g_{5/2}(z'e^{-x}) + g_{5/2}(z') + g_{5/2}(z'e^{-2x})\right)}{\left(g_{3/2}(z'e^{-x}) + g_{3/2}(z') + g_{3/2}(z'e^{-2x})\right)}}$$

c)

$x = \beta m_oH = \dfrac{m_oH}{kT}$

$x \to 0 \quad kT \gg m_oH \rightarrow$ las agitaciones térmicas predominan sobre el campo H

En el peso estadístico están poco separados los niveles de

$\varepsilon \Leftarrow \begin{cases}\varepsilon + m_oH\\ \varepsilon + 0\\ \varepsilon - m_oH\end{cases}$

$x \to \infty \quad kT \ll m_oH \rightarrow$ el campo domina sobre las fluctuaciones de T

$X \gg 1 \rightarrow$ $g_{3/2}(z' e^{-x}) \cong g_{3/2}(0) \rightarrow 0$

en la condensación será $z' = 1$

$g_{3/2}(z' e^{-2x}) \cong g_{3/2}(0) \rightarrow 0$, $g_{3/2}(z')$

$e^{\beta\mu} e^{x} = 1$

$$g_{3/2}(z') = g_{3/2}(z \cdot e^{x}) = \sum_{l=1}^{\infty} \frac{(z e^{x})^{l}}{l^{3/2}} \quad , \text{con } z = e^{\beta\mu}$$

$$\sum_{l=1}^{\infty} \frac{z^{l}}{l} e^{xl}$$

Condensación se hará en $X \gg 1 \rightarrow$

$e^{x} = e^{(-\beta\mu)}$ muy grande en valor absoluto

$X = \frac{-\mu}{kT_c}$

$T_c = \frac{-\mu}{k m_o H}$

$$N > N_{\varepsilon}^{MAX} = V\left(\frac{2\pi m k T_c}{h^2}\right)^{3/2} \left[g_{3/2}(z'=1) \right]$$

$$T_c^{3/2} = \frac{h^3 N}{V (2\pi m k)^{3/2}} \cdot \frac{1}{g_{3/2}(z'=1)} > T$$

$e^{\beta\mu} e^{x} = 1$

donde si $z' = 1$, como $x \gg 1 \rightarrow z^{-1} \gg 1 \rightarrow 1 \gg z$

$$T_c^{o\,3/2} = \frac{N}{V \cdot \underbrace{g_{3/2}(z=1)}_{2,612}} \cdot \frac{h^3}{(2\pi m k)^{3/2}}$$

$$\boxed{T_c^{3/2} = 2,612 \, (T_c^{o})^{3/2} \cdot \frac{1}{g_{3/2}(z'=1)}}$$

$T_c^{3/2} = (T_c^{o})^{3/2} \, 3$

$3 \frac{\lambda}{v} > \frac{g_{3/2}(1)}{}$

$$\lambda_c^{o\,3} = \frac{3 \cdot 2,612}{\langle m \rangle}$$

$$\lambda_c^{3} = \frac{1}{\langle m \rangle} \left[2.612 + e^{-x} + e^{-2x} \right]$$

$x \rightarrow$

$$\frac{\lambda^3}{v} = g_{3/2}(z) + \frac{z}{1-z}$$

$\frac{N T^{-3/2}}{V}$

$N T^{-3/2} V$

$N = 3N$

$N (3T^{-3/2})$

$3^{2/3} T$

4. gas de bosones con fase condensada $\Rightarrow$ $T<T_c$

$\langle N_0\rangle \ll \langle N\rangle$ $\rightarrow$ Tenemos algunas partículas cayendo en el nivel fundamental, pero los excitados están muchos más poblados.

$\langle E\rangle = \frac{3}{2}PV$ para toda situación ($3D$ y $\varepsilon = p^2/2m$)

$$U - TS + PV = N\mu$$

$$S = \frac{U - N\mu + PV}{T} \rightarrow S = \frac{3}{2}\frac{PV}{T} - \frac{N}{T}\mu + \frac{PV}{T} = \frac{5}{2}\frac{PV}{T} - \frac{N\mu}{T}$$

Tenemos dos fases

$$N_\varepsilon = N\left(\frac{T}{T_c}\right)^{3/2} \qquad N_0 = N - N_\varepsilon = N\left(1 - \left[\frac{T}{T_c}\right]^{3/2}\right)$$

$$\left.\begin{aligned} \frac{P}{kT} &= \frac{1}{\lambda^3} g_{5/2}(1) \\ \frac{N_\varepsilon}{V} &= \frac{1}{\lambda^3} g_{3/2}(1) = \frac{N - N_0}{V} \end{aligned}\right\}$$ ◄ Esto vale cuando tenemos las dos fases

$$ST = \frac{5}{2}PV - N\mu$$

$$\frac{P}{kT} = -\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}\,\ln\left(1 - z\,e^{-\beta\varepsilon}\right).d\varepsilon$$

$$\frac{P}{kT} = -\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}\,\ln\left(1 - z\,e^{-\beta\varepsilon}\right).d\varepsilon + \frac{1}{V}.\ln(1 - z)$$

↖ se va a cero en el límite termodinámico

con $z \to 1$ es $\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z=1)$

$$\frac{PV}{kT} = \frac{1}{\lambda^3} g_{5/2}(1)\,(N - N_0)\frac{\lambda^3}{g_{3/2}(1)} = (N - N_0)\frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$\frac{S}{k} = \frac{5}{2}(N - N_0)\frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{\mu}{kT}N$$

$z = e^{\beta\mu}$
$1 = e^{\beta\mu}$
$0 = \mu$

$$\frac{S}{k} = N\left[\left(1 - \frac{N_0}{N}\right)\frac{5}{2}\frac{g_{5/2}(1)}{g_{3/2}(1)}\right]$$

$$\boxed{S = kN_\varepsilon \frac{5}{2}\frac{g_{5/2}(1)}{g_{3/2}(1)}}$$

Podríamos pensar según que el efecto de la fase condensada es bajar la entropía pero esta ecuación tiene una "especie" de indeterminación

$$\underbrace{N}_{\to\infty}\underbrace{\left(1 - \frac{N_0}{N}\right)}_{\to 0}$$

y no nos da información

La entropía depende exclusivamente del # de partículas en los niveles excitados

5.

$$\frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) \quad \text{potencial OA 3D}$$

a)

$$R^2 = X^2 + Y^2$$

$$\varepsilon = \frac{m\omega^2 x^2}{2} + \frac{p^2}{2m}$$

$$Y^2 = \frac{p^2}{2m} \qquad \sqrt{2m}\, dY = dp$$

Sumamos sobre todos los estados

$$X^2 = \frac{m\omega^2 x^2}{2} \qquad dX = \sqrt{\frac{m}{2}}\,\omega\, dx$$

$$\sum_\varepsilon \doteq \sum_p \sum_x \longrightarrow \frac{1}{h}\iint_{-\infty}^{+\infty} dp\, dx \qquad \sqrt{\frac{2}{m}}\frac{dX}{\omega} = dx$$

$$\iint \frac{1}{h}\, dp\, dx = \iint \frac{1}{h}\sqrt{2m}\, dY \sqrt{\frac{2}{m}}\frac{1}{\omega}\, dX = \iint_{-\infty}^{+\infty} \frac{2}{h\omega}\, dY\, dX$$

Pasando a polares es: $dX\,dY = R\,d\theta\,dR \rightarrow 0<\theta<2\pi,\ 0<R<\infty$

$$= \iint \frac{2}{h\omega} R\, d\theta\, dR = \int 2\frac{2\pi}{h\omega} R\, dR = \frac{2}{\hbar\omega}\int_0^\infty R\, dR$$

pero $R = \varepsilon^{1/2}$ $\quad 2R\,dR = d\varepsilon$

$$\iint_{-\infty}^{+\infty} \frac{1}{h}\, dp\, dx = \frac{1}{\hbar\omega}\int_0^\infty d\varepsilon$$

En 1-D las energías son: $E = \hbar\omega\left(n + \frac{1}{2}\right)$ que corresponde a tener un factor $g(\varepsilon)$ constante porque serán los niveles equiespaciados.

b) en 3D se generaliza como sigue:

$$R^2 = X^2 + Y^2 + Z^2 + \alpha^2 + \gamma^2 + \eta^2$$

$$\varepsilon = \frac{m\omega_x^2 x^2}{2} + \frac{m}{2}\omega_y^2 y^2 + \frac{m}{2}\omega_z^2 z^2 + \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$\frac{1}{h^3}\iint_{-\infty}^{+\infty} d^3p\, d^3x \Rightarrow \text{para } X, Y, Z \text{ es } \sqrt{\frac{2}{m}}\frac{d\square}{\omega} = d\square$$

$$\alpha, \gamma, \eta \text{ es } \sqrt{2m}\, d\square = d\square$$

$$\frac{1}{h^3}\iiint\!\!\iiint_{-\infty}^{+\infty} (2m)^{1/2 \cdot 3}\, d\alpha\, d\gamma\, d\eta \left(\frac{2}{m}\right)^{3/2}\frac{1}{\omega^3}\, dX\, dY\, dZ$$

$$\frac{2^3}{(h\omega)^3}\iiint\!\!\iiint_{-\infty}^{+\infty} d\alpha\, d\gamma\, d\eta\, dX\, dY\, dZ$$

Pasando a esféricas en 6D es: $\quad \left(\frac{2}{h\omega}\right)^3 \pi^3 \cdot \frac{R^6}{6} = \frac{1}{(\hbar\omega)^3}\frac{R^6}{6}$ puesto que

$$V\left(\substack{\text{esfera } 6D\\ \text{de radio } R}\right) = \iint_{\text{ángulos}}\int_0^R r^5\, dr\, d\Omega = A\,\frac{R^6}{6} = \pi^3\frac{R^6}{6} \rightarrow A = \pi^3$$

$R = \varepsilon^{1/2}$

$2R\,dR = d\varepsilon$

$R\,dR = \frac{1}{2}\,d\varepsilon \rightarrow R^5\, dR = \frac{1}{2}\,d\varepsilon\, \varepsilon^2$

$$\frac{1}{(\hbar\omega)^3}\int_0^\infty R^5\, dR = \frac{1}{(\hbar\omega)^3}\int_0^\infty \frac{1}{2}\varepsilon^2\, d\varepsilon \Rightarrow \boxed{g(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3}\, d\varepsilon}$$

c)

$$Z_{GC} = \prod_{\varepsilon} \frac{1}{1 - e^{-\beta\varepsilon} \cdot z} = \prod_{\varepsilon} \frac{1}{1 - e^{-\beta(\varepsilon - \mu)}}$$

$$\Rightarrow \langle N \rangle = z \cdot \frac{\partial}{\partial z} \ln(Z_{GC}) = \sum_{\varepsilon} \frac{z e^{-\beta\varepsilon}}{1 - z e^{-\beta\varepsilon}} = \sum_{\varepsilon} \frac{1}{z^{-1} e^{\beta\varepsilon} - 1}$$

$$\langle N \rangle = \sum_{\substack{\varepsilon \\ \neq 0}} \frac{1}{z^{-1} e^{\beta\varepsilon} - 1} + \frac{1}{z^{-1} - 1}$$

Para un oscilador armónico en 3D es $\varepsilon = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right)$ con $n_x, n_y, n_z \in \mathbb{N}$

$\Rightarrow$ la energía del fundamental es: $\varepsilon_0 = \frac{3}{2}\hbar\omega$

$$\langle N_0 \rangle = \frac{1}{z^{-1} e^{\beta\varepsilon_0} - 1} = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}$$

si $\varepsilon_0 \sim \mu \rightarrow \langle N_0 \rangle$ se vuelve macroscópico.

d)

$$\langle N \rangle = \sum_{\varepsilon} \frac{1}{z^{-1} e^{\beta\varepsilon} - 1}$$

$$\sum_{\varepsilon} \longrightarrow \frac{1}{h^3} \iint d^3p \, d^3x \longrightarrow \frac{1}{2(\hbar\omega)^3} \int \varepsilon^2 \cdot d\varepsilon$$

$$\langle N \rangle = \frac{1}{2(\hbar\omega)^3} \int_0^{\infty} \frac{\varepsilon^2 \cdot d\varepsilon}{z^{-1} e^{\beta\varepsilon} - 1}$$

$\beta\varepsilon = x$, $\varepsilon = \frac{x}{\beta}$, $d\varepsilon = \frac{dx}{\beta}$

$$\langle N \rangle = \frac{1}{2(\hbar\omega)^3 \beta^3} \int_0^{\infty} \frac{x^2 \cdot dx}{z^{-1} e^{x} - 1}$$

NOTA: La adimensionalización aquí está a cargo de $1/h^3$

$$\langle N \rangle = \frac{1}{2(\hbar\omega\beta)^3} \Gamma(3) \, g_3(z) \rightarrow \langle N_{\varepsilon}^{MAX} \rangle = \frac{1}{\cancel{2}} \left(\frac{kT}{\hbar\omega} \right)^3 \cancel{2} \, g_3(z=1)$$

Hay Condensación de Bose para $N > N_\varepsilon^{MAX}$ $\Rightarrow$

$$N > N_\varepsilon^{MAX} = \left(\frac{kT}{\hbar\omega}\right)^3 g_3(1)$$

$$\frac{N(\hbar\omega)^3}{k^3 g_3(1)} > T^3 \Rightarrow \quad T_c^3 \equiv \left(\frac{\hbar\omega}{k}\right)^3 \frac{N}{g_3(1)}$$

La temperatura crítica será $\longrightarrow$ $\boxed{T_c = \frac{\hbar\omega}{k}\left[\frac{N}{g_3(1)}\right]^{1/3}}$

El límite termodinámico se compone de $\begin{cases} N\to\infty \\ V\to\infty \end{cases}$ con N/V constante. Aquí el $1/\omega^3$ hace las veces de volumen

$$\langle N\rangle = \frac{1}{2}\left(\frac{kT}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2 . dx}{z^{-1}e^x - 1}$$

$$\langle N\rangle = \frac{1}{z}\left(\frac{kT}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2 . dx}{z^{-1}e^x - 1} + \left(\frac{1}{z^{-1}-1}\right)$$

$$\langle N\rangle = \left(\frac{kT}{\hbar\omega}\right)^3 g_3(z) + \frac{z}{1-z} \rightarrow \text{el aporte del fundamental}$$

$$\left(\frac{\hbar}{kT}\right)^3 \omega^3 N = g_3(z) + \underbrace{\left(\frac{\hbar\omega}{kT}\right)^3}_{\to 0} \underbrace{\left(\frac{z}{1-z}\right)}_{\text{un número}}$$

en el límite termodinámico $\begin{cases} \omega^3 \to 0 \\ N\to\infty \end{cases}$ $N\omega^3 \to$ constante, $N_0 \to \infty$

$z \to 0$

$\frac{\omega^3}{1-z}$

$$\left(\frac{\hbar}{kT}\right)^3 \omega^3 N = g_3(z)$$

$$\left(\frac{\hbar}{kT}\right)^3 \# = g_3(z)$$

$\left(\frac{\hbar}{kT}\right)^3 \sim 0$ pues T alto

$z \sim 0$ pues $\begin{cases} \mu \to -\infty \\ T \text{ alto} \end{cases}$

$N \to \infty$
$\omega^3 \to \infty$
$\omega^3 \to 0$

Esto parece estar de acuerdo con lo que esperaríamos: con T altas estamos en comportamiento ideal.

e)

$$\ln(Z_{GC}) = \sum_\varepsilon \ln\left(\frac{1}{1-e^{-\beta\varepsilon}z}\right) = \frac{PV}{kT}$$

$$\frac{PV}{kT} = \frac{-1}{(\hbar\omega)^3 z}\int_0^\infty \varepsilon^2 . \ln\left(1-e^{-\beta\varepsilon}z\right). d\varepsilon$$

$$U = -\frac{\partial}{\partial\beta}\left(\ln Z_{GC}\right) = -\frac{\partial}{\partial\beta}\left(\frac{PV}{kT}\right) \Rightarrow$$

$$= \frac{1}{z(\hbar\omega)^3}\int_0^\infty \varepsilon^2 \left(\frac{1}{1-e^{-\beta\varepsilon}z}\right) + e^{-\beta\varepsilon}\varepsilon z . d\varepsilon$$

$$U = \frac{1}{z(\hbar\omega)^3}\int_0^\infty \frac{\varepsilon^3 . z . e^{-\beta\varepsilon}}{1-e^{-\beta\varepsilon}z} d\varepsilon$$

$$= \frac{\quad}{\quad}\int_0^\infty \frac{\varepsilon^3 . d\varepsilon}{z^{-1}e^{\beta\varepsilon} - 1}$$

$$U = \frac{1}{2(\hbar\omega)^3\beta^4}\int_0^\infty \frac{x^3\,dx}{z^{-1}e^x-1}$$

$$U = \frac{1}{2(\hbar\omega)^3\beta^4}\,\Gamma(4)\,g_4(z) = \boxed{\frac{3}{(\hbar\omega)^3\beta^4}\,g_4(z) = U}$$

Si $T<T_c \Rightarrow z\simeq 1 \rightarrow \quad U \cong \frac{3}{(\hbar\omega)^3}(kT)^4\, g_4(1)$ pero $T_c^3 = \left(\frac{\hbar\omega}{k}\right)^3 \frac{N}{g_3(1)}$

$$U \cong \frac{3N}{k^3 . g_3(1)}\frac{(kT)^4}{T_c^3}\, g_4(1)$$

$$\boxed{\frac{U}{k.T_c.N} \cong 3\left(\frac{T}{T_c}\right)^4 \frac{g_4(1)}{g_3(1)}}$$ ◄ Energía en función de la temperatura crítica

6.

El sólido se analiza pensando en términos de fonones → cumplen bose-einstein statistic. Los fonones tienen $\mu=0$ porque No necesito U para crearlos o destruirlos.

Un sólido se puede pensar como con un: $H = \phi_0 + \sum_{i=1}^{3N}\frac{1}{2}\left(m\dot{q}_i^2+\omega_i^2 q_i^2\right)$ ← en modos normales

$$\Rightarrow U = \phi_0 + \sum_i^{3N}\left(n_i+\frac{1}{2}\right)\hbar\omega_i$$

$$U = \phi_0 + \sum_i^{3N}\frac{1}{2}\hbar\omega_i + \sum_i^{3N} n_i\hbar\omega_i \rightarrow \sum_i^{3N}\left(\frac{1}{e^{\beta\hbar\omega_i}-1}\right)\hbar\omega_i$$

a) Debye plantea:

$$3N = \int_0^{\omega_D} \underbrace{g(\omega)\,d\omega}$$ → # de modos normales entre $(\omega,\omega+d\omega)$

sea, en 3D;

$\omega = c.k$ dispersión

$$k_x L = n_x\pi \rightarrow n_x = \frac{k_x.L}{\pi}$$

Descompongo en 3N osciladores independientes

$\frac{1}{8}$ de esfera en $n_x n_y n_z$ ← $n^2 = n_x^2+n_y^2+n_z^2 = 3n_x^2 = 3\frac{k_x^2L^2}{\pi^2}$

$$n_x^3 = \left(\frac{k_xL}{\pi}\right)^3 \rightarrow \cancel{3}n_x^2\,dn_x = \cancel{3}\frac{k_x^2L^3}{\pi^3}\,dk_x$$

$$\#_{modos} = \int \frac{1}{8}4\pi . n^2\,dn \rightarrow$$

$$\frac{1}{8} . 4\cancel{\pi}\frac{\omega^2}{c^2}\frac{V}{\pi^{\cancel{3}2}}.dk = \frac{1}{8}\frac{4V\omega^2}{c^3\pi^2}\,d\omega$$

Calculamos la cantidad de modos normales en un sólido de volumen $(L^3)=V$ pidiendo condiciones periódicas de contorno.

$$3N = \int_0^{\omega_D}\frac{V\omega^2}{2c^3\pi^2}\,d\omega = \frac{V}{c^3\pi^2}\frac{\omega_D^3}{6}$$

$$\omega_D^3 = \frac{18c^3\pi^2 N}{V}$$

$$\omega_D = \left(\frac{N\pi^2 18}{V}\right)^{1/3} c$$

Para D dimensiones, con:

$$\omega = \alpha k^s$$ — dimensiones del problema

$$k_i L = n_i \pi \longrightarrow n^2 = \sum_{i=1}^{D} n_i^2 \rightarrow n^2 = D \cdot n_i^2$$

$$n_i^2 = \left(\frac{k_i L}{\pi}\right)^2 = \left(\frac{\omega^{1/s} L}{\alpha^{1/s} \pi}\right)^2$$

$$n^2 = D \cdot \frac{\omega^{2/s} L^2}{\alpha^{2/s} \pi^2} \rightarrow$$

donde A es la integral sobre la parte angular

$$n_i^D = \frac{\omega^{D/s} L^D}{\alpha^{D/s} \pi^D} \rightarrow$$

$$\cancel{D} n_i^{D-1} dn_i = \frac{L^D}{\alpha^{D/s} \pi^D} \omega^{\frac{D}{s}-1} \frac{\cancel{D}}{s} d\omega$$

$$\#_{\text{modos de excitación}} = \frac{1}{2^D} A \int n^{D-1} dn = A \int \frac{L^D}{(2\pi)^D} \frac{1}{\alpha^{D/s} \cdot s} \omega^{\frac{D}{s}-1} d\omega$$

$$\Rightarrow \boxed{g(\omega) d\omega = \frac{L^D}{(2\pi)^D} A \frac{1}{s \cdot \alpha^{D/s}} \omega^{\frac{D-s}{s}} d\omega}$$

$\epsilon = \hbar\omega$
$d\epsilon = \hbar\, d\omega$

con $D \cdot N = \int_0^{\omega_D} g(\omega) d\omega$

Pasamos al continuo la parte que depende de T en U:

$$U = \int_0^{\omega_D} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} d\omega$$

b)

$$C_V = \left.\frac{\partial U}{\partial T}\right|_V = \sum_{i=1}^{DN} \hbar\omega_i \cdot -1 \cdot (e^{\beta\hbar\omega_i} - 1)^{-2} e^{\beta\hbar\omega_i} \frac{\hbar\omega_i (-1)}{k \; T^2}$$

$$= \sum_{i=1}^{DN} \frac{(\hbar\omega_i)^2 e^{\beta\hbar\omega_i}}{[e^{\beta\hbar\omega_i} - 1]^2} \frac{1}{kT^2}$$

$\frac{D}{s} - 1 + 2$

$$= \int_0^{\omega_D} \left(\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right)^2 \frac{e^{\beta\hbar\omega}}{k T^2} \frac{L^D}{(2\pi)^D} A \frac{\omega^{\frac{D-s}{s}}}{s \cdot \alpha^{D/s}} d\omega$$

$$= \frac{L^D}{(2\pi)^D} A \frac{1}{s \cdot \alpha^{D/s}} k \int_0^{\omega_D} \left(\frac{\hbar}{kT}\right)^2 \omega^{\frac{D+s}{s}} e^{\frac{\hbar\omega}{kT}} \frac{d\omega}{(e^{\frac{\hbar\omega}{kT}} - 1)^2}$$

$\frac{\hbar\omega}{kT} = x$
$d\omega = dx \frac{kT}{\hbar}$

$$C_V = \frac{L^D A\, k}{(2\pi)^D s \cdot \alpha^{D/s}} \cancel{\frac{kT}{\hbar}} \left(\frac{\hbar}{kT}\right)^{\cancel{2}} \left(\frac{kT}{\hbar}\right)^{\frac{D+s}{s}} \int_0^{\frac{\hbar\omega_D}{kT}} \frac{x^{\frac{D+s}{s}} e^x dx}{(e^x - 1)^2}$$

$$C_V = \frac{L^D A\, k}{(2\pi)^D \cdot s \cdot \alpha^{D/s}} \left(\frac{kT}{\hbar}\right)^{\frac{D}{s}} \underbrace{\int_0^{\hbar\omega_D/kT} \frac{x^{\frac{D+s}{s}} e^x dx}{(e^x - 1)^2}}_{\equiv I}$$

$$dV = \frac{e^x}{(e^x-1)^2} \qquad U = x^{\frac{D+S}{S}}$$
$$du = \left(\frac{D+S}{S}\right) x^{\frac{D}{S}}\, dx$$

$$V = \int \frac{e^x}{(e^x-1)^2}\, dx$$

$e^x - 1 = z$
$e^x dx = dz$
$dx = e^{-x} dz$
$dx = \frac{1}{z+1} dz$

$$V = \int \frac{dz}{z^2} = -\frac{1}{z}$$

$$I = \frac{-x^{\frac{D+S}{S}}}{(e^x-1)}\Bigg|_0^{\frac{\hbar\omega_D}{kT}} + \int_0^{\frac{\hbar\omega_D}{kT}} \frac{x^{\frac{D}{S}}}{e^x-1}\, dx \left(\frac{D+S}{S}\right)$$

$$I = \qquad + \left(\frac{D+S}{S}\right)\int_0^{\hbar\omega_D/kT}$$

Ahora, en el límite $T \lll 1 \Rightarrow \frac{\hbar\omega}{kT} = x \ggg 1 \Rightarrow$ podemos considerar $x \to \infty$

$$I \simeq \left(\frac{D+S}{S}\right) \int_0^\infty \frac{x^{D/S}}{e^x-1}\, dx \quad \therefore$$

I será un # que no dependerá de la temperatura, con lo cual:

$$C_V \approx \frac{L^D . A . k}{(2\pi)^D\, S . \alpha^{D/S}} \frac{k}{\hbar} T^{D/S} . I \quad \rightarrow \quad \boxed{C_V \propto T^{D/S}}$$

c) Para $T \ggg 1 \rightarrow x \lll 1 \Rightarrow$

$$I \cong \frac{-x^{\frac{D+S}{S}}}{+x}\Bigg|_0^{\frac{\hbar\omega_D}{kT}} + \int_0^{\frac{\hbar\omega_D}{kT}} x^{\frac{D}{S}-1}\, dx \left(\frac{D+S}{S}\right)$$

$$I \cong -x^{D/S}\Big|_0^{\frac{\hbar\omega_D}{kT}} + \left(\frac{D+S}{S}\right)\frac{1}{\left(\frac{D}{S}\right)} x^{D/S}\Big|_0^{\frac{\hbar\omega_D}{kT}}$$

$$\left(\frac{S}{D}\right) x^{D/S}\Big|_0^{\hbar\omega_D/kT} = \left(\frac{S}{D}\right)\left(\frac{\hbar\omega_D}{kT}\right)^{D/S}$$

$$C_V = \frac{L^D . A . k}{(2\pi)^D\, S . \alpha^{D/S}} \cancel{\left(\frac{kT}{\hbar}\right)^{D/S}} \left(\frac{S}{D}\right)\left(\frac{\cancel{\hbar}\,\omega_D}{\cancel{kT}}\right)^{D/S}$$

$$C_V = \frac{L^D . A . k}{(2\pi)^D\, S . \alpha^{D/S}} \left(\frac{S}{D}\right) \omega_D^{D/S}$$

- C_V no depende de T en el límite de temperaturas altas porque tiende al valor clásico $C_V = 3Nk$ (en 3D con la dispersión usual).
- Asimismo tampoco depende de $\hbar$ porque esto denunciaría comportamiento cuántico.

$$D.N = \frac{L^D}{(2\pi)^D} A \frac{1}{S\alpha^{D/S}} \omega_D^{\frac{D}{S}} . \frac{S}{D} \rightarrow \boxed{C_V = D.N.k}$$ glorioso !!!