Serie 6: Gas de Fermi

1.

energía de la partícula → 0, ε, 2ε

3 niveles de energía (0,1,2); dos partículas

Sistema en baño térmico a temperatura T

a) función de partición: Z

ii) Estadística de Bose-Einstein

E	nivel
2ε	2
ε	1
0	0

$$Z_c = \sum_{\{n_\epsilon\}} e^{-\beta E_\epsilon n_\epsilon}$$ con el constraint → $\sum_\epsilon n_\epsilon = N$

$2 = n_0 + n_1 + n_2$ ⇒

n_0	n_1	n_2	ϵ_0	ϵ_1	ϵ_2	E
2	0	0	0			0
0	2	0		2ε		2ε
0	0	2			4ε	4ε
1	1	0	0	ε		ε
0	1	1		ε	2ε	3ε
1	0	1	0		2ε	2ε

$$Z_c = \left(\sum_{n_0=0}^{2} e^{-\beta.0.n_0}\right)\left(\sum_{n_1=0}^{2} e^{-\beta.\varepsilon.n_1}\right)\left(\sum_{n_2=0}^{2} e^{-\beta.(2\varepsilon).n_2}\right) = \sum_{n_0,n_1,n_2=0}^{2} e^{-\beta(\epsilon_0 n_0 + \epsilon_1 n_1 + \epsilon_2 n_2)}$$

ϵ_n = energía correspondiente al # de ocupación n

$$Z_c = e^{-\beta\epsilon_0 2} + e^{-\beta\epsilon_1 2} + e^{-\beta\epsilon_2 2} + e^{-\beta(\epsilon_0 1 + \epsilon_1 1)} + e^{-\beta(\epsilon_1 1 + \epsilon_2 1)} + e^{-\beta(\epsilon_0 .1 + \epsilon_2 .1)}$$

$$Z_c = 1 + e^{-2\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-\beta 3\varepsilon} + e^{-\beta 2\varepsilon}$$

$$\boxed{Z_c = 1 + 2e^{-2\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon}}$$

iii) Estadística de Fermi-Dirac

n_0	n_1	n_2	ϵ_0	ϵ_1	ϵ_2	
1	1	0	-	ε	-	ε
0	1	1	-	ε	2ε	3ε
1	0	1	0	-	2ε	2ε

$$Z_c = e^{-\beta(\epsilon_0 1 + \epsilon_1 1)} + e^{-\beta(\epsilon_1 .1 + \epsilon_2 .1)} + e^{-\beta(\epsilon_0 .1 + \epsilon_2 1)}$$

$$\boxed{Z_c = e^{-\beta.\varepsilon} + e^{-3\beta.\varepsilon} + e^{-2\beta\varepsilon}}$$

i) Estadística de Maxwell-Boltzmann

Ahora consideramos a las partículas como distinguibles; con lo cual podemos etiquetarlas: partícula A y partícula B

ϵ_0	ϵ_1	ϵ_2	
A,B	0	0	0
0	A,B	0	2ε
0	0	A,B	4ε
A	B	0	ε
B	A	0	ε
A	0	B	2ε
B	0	A	2ε
0	A	B	3ε
0	B	A	3ε

$$\boxed{Z_c = \frac{1}{2}\left[1 + 3 e^{-2\beta\varepsilon} + 2 e^{-\beta\varepsilon} + 2 e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon} \right]}$$

b) La energía media será

$$\langle E \rangle = \frac{\sum_\nu e^{-\beta E_\nu} E_\nu}{\sum_\nu e^{-\beta E_\nu}} \Rightarrow$$

$$Z_c^{(MB)} = \frac{1}{2}\left(1 + 3\left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{2} + 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) = \frac{49}{32}$$

$$Z_c^{(fd)} = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 = \frac{7}{8}$$

$$Z_c^{(Be)} = 1 + 2\left(\frac{1}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{35}{16}$$

$$\langle E \rangle^{(MB)} = \frac{32}{49}\left[3 \cdot 2\varepsilon \left(\frac{1}{2}\right)^2 + 4\varepsilon \left(\frac{1}{2}\right)^4 + 2\varepsilon \cdot \frac{1}{2} + 2 \cdot 3\varepsilon \left(\frac{1}{2}\right)^3 \right]$$

$$\langle E \rangle^{(MB)} = \frac{32}{49} \varepsilon \left(\frac{7}{4}\right) = \frac{8}{7}\varepsilon \sim 1{,}14 \, \varepsilon$$

$$\langle E \rangle^{(fd)} = \frac{8}{7} \left[\varepsilon \cdot \frac{1}{2} + 3\varepsilon \left(\frac{1}{2}\right)^3 + 2\varepsilon \left(\frac{1}{2}\right)^2 \right]$$

$$= \frac{8}{7} \varepsilon \left(\frac{11}{8}\right) = \frac{11}{7}\varepsilon \sim 1{,}57 \, \varepsilon$$

$$\langle E \rangle^{(Be)} = \frac{16}{35} \left[2 \cdot 2\varepsilon \left(\frac{1}{2}\right)^2 + \varepsilon \left(\frac{1}{2}\right) + 3\varepsilon \left(\frac{1}{2}\right)^3 + 4\varepsilon \left(\frac{1}{2}\right)^4 \right]$$

$$\langle E \rangle^{(Be)} = \frac{\cancel{16}^{\,2}}{35} \varepsilon \, \frac{17}{\cancel{8}_{\,1}} = \frac{34}{35} \varepsilon \sim 0{,}97 \, \varepsilon$$

2.

$10^7\,K$ — T de la enana blanca

$\frac{N}{V} = n \sim 10^{10}\,\mathrm{Kg/m^3}$ densidad aproximada de la enana blanca

* Tendremos comportamiento altamente degenerado cuando la densidad sea alta y/o la temperatura baja.

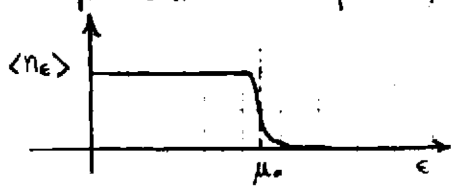
Los átomos de He se hallarán totalmente ionizados si la energía del «baño» es mucho mayor que su energía de ionización:

$$kT \gg \epsilon_{ionización}$$

$$8{,}6 \cdot 10^{-5}\,\frac{eV}{K} \cdot 1 \cdot 10^{7}\,K = 860\,eV \gg 10\,eV$$

$\Rightarrow$ están totalmente ionizados

* Un gas de fermiones puede considerarse como hallándose a temperatura nula si el # de ocupación medio tiene un comportamiento tipo "escalón":



Esto se cumple cuando: $T \ll T_F \equiv \frac{\mu_0}{k}$ o bien $\frac{\lambda^3}{v} \ggg 1$

$$N = \int_0^{\epsilon_F} \frac{4\pi \cdot V}{h^3} m^{3/2} \sqrt{2}\,\sqrt{\varepsilon}\,.d\varepsilon$$

expresión para:
- problema en 3D
- relación $\epsilon = p^2/2m$

$$N = \frac{4\pi \cdot V}{h^3} m^{3/2}\sqrt{2}\,\frac{\varepsilon^{3/2}}{3/2}\Big|_0^{\epsilon_F}$$

$$\frac{3 \quad h^3 N}{\sqrt{2} \cdot 8\pi \; m^{3/2}\, V} = \epsilon_F^{3/2}$$

$$\left(\frac{h^3 \; 3N}{2^{3/2}\,\pi\; m^{3/2}\, V}\right)^{2/3} = \epsilon_F$$

$\rho = 10^{10}\,\frac{kg}{m^3}$

$\frac{N.m}{V} = \rho$

$=$

$$\frac{h^2}{m} \cdot \left(\frac{3N}{\pi V}\right)^{2/3} \frac{1}{2^{7/3}}$$

$$2 \cdot 2^{4/3} = 2 \cdot 4^{2/3}$$

$$\epsilon_F = \frac{h^2}{2m} \cdot \left(\frac{3N}{4\pi V}\right)^{2/3}$$

$$\epsilon_F = \frac{h^2}{2m}\left(\frac{3\rho}{4\pi m}\right)^{2/3} = 2{,}1 \cdot 10^{-37}\; 1{,}9 \cdot 10^{26}\; \frac{1}{\cancel{m^2}} \cdot \frac{m^{\cancel{4}\,2}\,\cancel{kg^2}}{\cancel{kg}\; s^{\cancel{4}\,2}} \cdot \cancel{s^2}$$

$$\epsilon_F = 1{,}5 \cdot 10^{-11}\,J$$

$$1 \ll \frac{T_F}{T} = 300.000 \longleftarrow T_F \equiv \frac{\epsilon_F}{k} = \frac{\mu_0}{k} = \frac{1{,}5 \cdot 10^{-11}\,J}{1{,}38 \cdot 10^{-23}\,J/K} \sim 3 \cdot 10^{12}\,K$$

$\Rightarrow$ Podemos considerar el gas como a $T = 0$

3.

degeneración del nivel (# de ocupación)

a) La energía de Fermi saldrá de: $N = \int_0^{\epsilon_F} d\varepsilon \cdot g \cdot g(\varepsilon)$ donde $g(\varepsilon)$ es un factor de forma que surge de pasar la $\int dp$ a $\int d\varepsilon$.

$$\sum_p^{\infty} \rightarrow \frac{V}{h^3}\int_{-\infty}^{\infty} d^3p = \frac{V}{h^3}\int d\theta\, d\varphi \operatorname{sen}\theta\, p^2\, dp$$ pues hay simetría esférica $p^2 = p_x^2 + p_y^2 + p_z^2$

obs.

$N = \sum_p \langle n_p \rangle$

sumamos sobre los autoestados de momento $\Rightarrow$ al continuo

$$N = \frac{V}{h^3}\int_{-\infty}^{\infty} d^3p \,\langle n_p\rangle$$

$$N = \int_{-\infty}^{+\infty} \frac{V}{h^3} \cdot d^3p\, \underbrace{\langle n_p \rangle}_{1 \text{ para } p<p_F}$$

$$= \frac{V}{h^3}4\pi \int_0^{p_F} p^2 \cdot dp \qquad \text{para} \quad p = \sqrt{2m\varepsilon}$$

$$= \frac{4\pi V}{h^3}\int_0 \sqrt{2m\varepsilon}\, m\, d\varepsilon \qquad p \cdot dp = m \cdot d\varepsilon$$

$$= \frac{4\pi V \sqrt{2}\, m^{3/2}}{h^3}\int_0^{\epsilon_F} \varepsilon^{1/2}\, d\varepsilon$$

$$N = \frac{4\pi V \sqrt{2}\, m^{3/2}}{h^3}\frac{\epsilon_F^{3/2}}{3/2}\cdot g \qquad , \text{con } g = \begin{cases} 1 & \text{part. sin spin} \\ 2 & \text{part. con spin} \end{cases}$$

$$\epsilon_F^{3/2} = \frac{3h^3 N}{g\, 8\pi V \sqrt{2}\, m^{3/2}}$$

$$\epsilon_F = \left(\frac{3h^3 N}{8\pi V g\, 2^{1/2} m^{3/2}}\right)^{2/3} = \frac{h^2}{m}\left(\frac{3N}{4\pi g V}\right)^{2/3}\frac{1}{(2^{1/2}\cdot 2)^{2/3}}$$

, donde g es la degeneración correspondiente en general a un grado de libertad interno $\longrightarrow$ $\boxed{\epsilon_F = \left(\frac{3N}{4\pi g V}\right)^{2/3}\frac{h^2}{2m}}$

b)

$$\langle E \rangle = \sum_i \langle n_i \rangle \epsilon_i$$

$$\langle E \rangle = \sum_p \langle n_p \rangle \frac{p^2}{2m}$$

integrado en esféricas

$$\sum_p \frac{p^2}{2m} \xrightarrow{\text{PASO AL CONTINUO}} \int_{-\infty}^{+\infty} \frac{V}{h^3}\frac{p^2}{2m}d^3p = \frac{V}{h^3 2m}4\pi\int_0^{\infty} p^4 \cdot dp \qquad p = (2m\varepsilon)^{1/2}, \quad p \cdot dp = m \cdot d\varepsilon$$

$$= \frac{V 4\pi}{h^3 2m}\int_0^{\infty} (2m\varepsilon)^{3/2} m \cdot d\varepsilon$$

$$= \frac{4\pi V}{2h^3}(2m)^{3/2}\int_0^{+\infty} \varepsilon^{3/2} \cdot d\varepsilon \Rightarrow$$

ahora utilizamos la distribución de Fermi para $T=0$ (es decir $\langle n_i \rangle = 1$ si $\epsilon < \epsilon_F$, $\langle n_i \rangle = 0$ si $\epsilon > \epsilon_F$) $\Rightarrow$

$$E(T=0) = \frac{4\pi V \sqrt{2}\, m^{3/2}}{h^3}\int_0^{\epsilon_F} \epsilon^{3/2} d\varepsilon = \frac{4\pi V \sqrt{2}\, m^{3/2}}{h^3}\frac{1}{5/2}\epsilon_F^{5/2}$$

$$E_0 = \frac{8\pi V \sqrt{2}\, m^{3/2}}{5h^3}\epsilon_F^{5/2}$$

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$$\frac{E_0}{N} = \left(\frac{8\pi V\, 2^{1/2} m^{3/2}}{3h^3 N}\right)\frac{3}{5}\epsilon_F^{5/2}$$

$$\boxed{E_0 = \frac{3}{5}N\epsilon_F} \longleftarrow \frac{E_0}{N} = \epsilon_F^{-3/2}\frac{3}{5}\epsilon_F^{5/2} = \frac{3}{5}\epsilon_F$$

• **NOTA**: Hemos considerado a los electrones sin spin ($g=1$)

c)

$$E = \frac{4\pi V}{2h^3}(2m)^{3/2}\int_0^{+\infty} \varepsilon^{3/2}\cdot\frac{1}{\left(e^{\frac{\varepsilon-\mu}{kT}}+1\right)}\,d\varepsilon$$

$$\frac{P}{kT} = \frac{1}{\lambda^3} f_{5/2}(z) = \frac{(2\pi m kT)^{3/2}}{h^3}\cdot\sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}}(-1)^{l+1}$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\left[\ln(Z_{GC})\right] = -\frac{\partial \ln Z_{GC}}{\partial T}\cdot\frac{\partial T}{\partial\beta} = kT^2\frac{\partial \ln Z_{GC}}{\partial T}(z,V,T)$$

$$\beta=\frac{1}{kT}\rightarrow T=\frac{1}{k\beta}$$

$$\frac{dT}{d\beta}=\frac{1}{k}\cdot\frac{-1}{\beta^2}=\frac{-1\,k^2T^2}{k}=-kT^2$$

$$\langle E\rangle = kT^2\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)$$

$$\langle E\rangle = kT^2\cdot\frac{\partial}{\partial T}\left(\frac{V}{\lambda^3}f_{5/2}(z)\right)$$

$$\langle E\rangle = kT^2V\left[f_{5/2}(z)\cdot\frac{\partial}{\partial T}\left(1/\lambda^3\right)\right] = kT^2V\cdot\frac{3}{2}\frac{f_{5/2}(z)}{\lambda^3 \cancel{T}} = \left(\frac{kT}{\lambda^3}f_{5/2}(z)\right)\frac{3}{2}V$$

$$\boxed{E = P\frac{3}{2}V}$$

$$\frac{\partial}{\partial T}\left(\frac{1}{\lambda^3}\right) = \frac{(2\pi m k)^{3/2}}{h^3}\frac{3}{2}T^{1/2} = \frac{3}{2}\cdot\frac{1}{\lambda^3}\cdot\frac{1}{T}$$

• **Nota importante**:
La función de partición $Z_{GC}=Z_{GC}(z,T,V)$ y para los cálculos de $\langle E\rangle$ no consideramos que $z=z(T)$.

A $T=0$ será:
$$P(T=0)=P_0=\frac{2}{3}\frac{E_0}{V}=\frac{2}{\cancel{3}V}\frac{\cancel{3}}{5}N\cdot\epsilon_F$$

$$\boxed{P_0=\frac{2}{5}\cdot\frac{N}{V}\cdot\epsilon_F}$$

4.
Gas de electrones en 2-D sobre un área A

a y b)
$$\sum_p \rightarrow \frac{A}{h^2}\int_{-\infty}^{+\infty}d^2p = \frac{A}{h^2}\iint_{-\infty\,-\infty}^{+\infty\,+\infty}dp_x\,dp_y = \frac{A}{h^2}\int_0^{2\pi}\int_0^{\infty}p\cdot d\theta\cdot dp = \frac{2\pi A}{h^2}\int_0^{\infty}p\cdot dp$$

$$= \frac{2\pi A}{h^2}\int_0^{\infty}m\cdot d\varepsilon = \frac{2\pi A m}{h^2}\int_0^{+\infty}d\varepsilon$$

metemos la distribución de Fermi

$$N = \frac{2\pi A m}{h^2}\cdot g\int_0^{\infty}\frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

$$\sum_p\frac{p^2}{2m}\rightarrow\frac{A}{h^2}\int_{-\infty}^{+\infty}\frac{p^2}{2m}\cdot d^2p = \frac{A}{2mh^2}\int_0^{+\infty}\int_0^{2\pi}p^3\,d\theta\cdot dp = \frac{2\pi A}{2mh^2}\int_0^{\infty}p^3\cdot dp = \frac{2\pi A}{2mh^2}\int_0^{\infty}2m^2\varepsilon\cdot d\varepsilon$$

$$\langle E\rangle = \frac{2\pi A m}{h^2} g\int_0^{\infty} d\varepsilon\ \varepsilon\ F(\varepsilon)$$

$$\langle N\rangle = \frac{2\pi A\ m}{h^2}\ g\int_0^{\infty}\frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

$$\langle E\rangle = \frac{2\pi A m}{h^2}.g\int_0^{+\infty}\frac{\varepsilon\ d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

a $T=0$ es

$$\langle N\rangle = \frac{2\pi A m}{h^2}\ g\int_0^{\epsilon_F} d\varepsilon$$

Este $\langle N\rangle$ es válido para todo T pues ya que ha sido calculado con T=0

pero $\langle E\rangle = -\frac{\partial}{\partial\beta}\left(\ln Z_{GC}\right)$ $\rightarrow Z_{GC}(z,T,A)$

$$\langle N\rangle = \frac{2\pi A m.g}{h^2}\ \epsilon_F = N(T=0)$$

$$-\frac{\partial}{\partial\beta}\left(\frac{P.A}{kT}\right)$$

$$-\frac{\partial}{\partial T}\left(\frac{P.A}{kT}\right)\frac{\partial T}{\partial\beta}$$

$$\boxed{\epsilon_F = \frac{\langle N\rangle h^2}{2\pi A m g}}$$

$$\langle E\rangle = kT^2\ \frac{\partial}{\partial T}\left(\frac{P.A}{kT}\right)$$

energía de Fermi en términos del # de partículas medio

$$\langle E\rangle = kT^2\frac{\partial}{\partial T}\left(\frac{g}{\lambda^2}f_{5/2}(z)\right)A$$

$$= kT^2 A\ g\ f_{5/2}(z)\ \frac{\partial}{\partial T}(\lambda^{-2}) = kT^2 A\ g.\frac{2\pi m k}{h^2}\ f_{5/2}(z)$$

$$\frac{1}{\lambda^2} = \frac{(2\pi m\ kT)}{h^2} \rightarrow \frac{\partial}{\partial T}\left(\frac{1}{\lambda^2}\right) = \frac{2\pi m k}{h^2}$$

$$= kTA\ g\ \frac{2\pi m k T}{h^2}\ f_{5/2}(z)$$

$$\langle E\rangle = \cancel{kT}A.\frac{P}{\cancel{kT}} \rightarrow \boxed{\langle E\rangle = P.A}$$

$$\frac{P.A}{kT} = \frac{\langle E\rangle}{kT} = \frac{2\pi A m.g}{h^2 kT}\int_0^{\infty}\frac{\varepsilon.\ d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

$$\boxed{\begin{array}{ll}\frac{\varepsilon}{kT}=x & d\varepsilon = kT.dx\\ \frac{\mu}{kT}=\xi & \\ \text{auxiliar} & \end{array}}$$

$$\frac{P.A}{kT} = \frac{2\pi A m}{h^2 kT}.g\int_0^{\infty}\frac{(kT)^2 x\ dx}{e^{x-\xi}+1}$$

con $T\sim 0$, $T\ll T_F$ considero al sistema altamente degenerado

$$\frac{P.A}{kT} = \frac{2\pi A m g kT}{h^2}\left[\int_0^{\xi}x.dx + \frac{\pi^2}{6}\right]$$

$$\frac{P.A}{kT} = \frac{2\pi A m g kT}{h^2}\left(\frac{1}{2}\xi^2 + \frac{\pi^2}{6}\right)$$

$$T \ll \frac{\epsilon_F}{k}$$

$$\frac{P.A}{kT} = \frac{2\pi A\ m}{h^2}\ g\ kT\left(\frac{\mu^2}{2(kT)^2}+\frac{\pi^2}{6}\right)$$

expresión para $P.A/kT$ en función de T,μ $\longrightarrow$

$$\boxed{\frac{P.A}{kT} = \frac{2\pi A m g}{h^2}\left(\frac{\mu^2}{2kT}+\frac{kT\pi^2}{6}\right)}$$

c)

$$\langle N\rangle = \frac{2\pi A m}{h^2}g\int_0^{\infty}\frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} \qquad \left(e^{\frac{\varepsilon}{kT}}z^{-1}+1\right)^{-1}$$

$$\langle N\rangle = A.g.\frac{2\pi m kT}{h^2}\int_0^{\infty}\frac{dx}{e^{x}.z^{-1}+1} \rightarrow f_1(z).\overbrace{\Gamma(\nu=1)}^{=1}$$

$$\frac{\varepsilon}{kT}=x \qquad d\varepsilon = kT.dx$$

$$\frac{\langle N\rangle}{A} = \frac{g}{\lambda^2}\ .\ f_1(z)$$

$$\rightarrow \sum_{\ell=1}^{\infty}\frac{z^{\ell}}{\ell}(-1)^{\ell+1}$$

$$\varepsilon = kT.x \qquad d\varepsilon = kT.dx$$

$$\frac{PA}{kT} = \frac{2\pi A m g}{h^2}\frac{1}{kT}\int_0^\infty \frac{\varepsilon . d\varepsilon}{e^{\varepsilon/kT}z^{-1}+1}$$

$$(kT)^2 \int_0^\infty \frac{x.dx}{e^x z^{-1}+1}$$

$$\frac{P.A}{kT} = A g \frac{2\pi m kT}{h^2} f_2(z).\Gamma(2)$$

$$\frac{PA}{kT} = A.\frac{g}{\lambda^2}. f_2(z).\Gamma(2), \text{ con } f_2(z) = \sum_{l=1}^{\infty}\frac{z^l}{l^2}(-1)^{l+1}$$

$$\frac{\partial f_2(z)}{\partial z} = \sum_{l=1}^{\infty} \cancel{l}\frac{z^{l-1}}{l^{\cancel{2}}}(-1)^{l+1}$$

$$\ln\left[\prod_j (1+e^{-\beta\varepsilon_j}z)\right] = \ln Z_{GC} = \frac{PA}{kT} = \frac{A}{z}\frac{\langle N\rangle}{A}$$

$$\frac{\partial f_2(z)}{\partial z} = \frac{1}{z}\sum_{l=1}^{\infty}\frac{z^l}{l}(-1)^{l+1}$$

$$f_2(z) = \int \frac{1}{z}.f_1(z).dz$$

En el caso de $f_1(z)$ la integral puede hacerse inmediatamente como:

$$= \int_0^\infty + \frac{d}{dx}\left(-\ln\left[e^{-x}z+1\right]\right) = -\ln\left(e^{-x}z+1\right)\Big|_0^\infty = \ln\left(1+e^{-x}z\right)^{-1}\Big|_0^\infty$$

$$0 - \ln\left(\frac{1}{z+1}\right)$$

idea

$$\boxed{\frac{\partial}{\partial x}\left(\ln\left(e^{-x}z+1\right)\right) = \frac{1}{e^{-x}z+1}\left(-e^{-x}z\right) = -\frac{-1}{1+e^{x}z^{-1}}}$$

$$\frac{\langle N\rangle}{A} = \frac{g}{\lambda^2}.-\ln(z+1)^{-1}$$

$$\varepsilon_F \frac{2\pi m g}{h^2} = \frac{g}{\lambda^2}.-\ln(z+1)^{-1}$$

$$-\beta\varepsilon_F \frac{\cancel{g}}{\cancel{\lambda^2}} = \frac{\cancel{g}}{\cancel{\lambda^2}}.\ln(z+1)^{-1}$$

$$e^{-\beta\varepsilon_F} = \frac{1}{z+1}$$

$$z.e^{-\beta\varepsilon_F} = 1 - e^{-\beta\varepsilon_F} \; !$$

$$-\beta\varepsilon_F + \mu\beta = \ln(1-e^{-\beta\varepsilon_F}) \longrightarrow \boxed{\mu = \varepsilon_F + \frac{1}{\beta}.\ln\left(1-e^{-\beta\varepsilon_F}\right)}$$

d)

$$C_V = \frac{\partial E}{\partial T}\Big|_{V,N} = \frac{3}{2}V\left(\frac{\partial P}{\partial T}\right)_{V,N} \Leftarrow$$ Para 3D vale esto, acá es $C_V = \frac{\partial(PA)}{\partial T}\Big|_{A,N} = A.\frac{\partial P}{\partial T}$

con el sistema altamente degenerado es $T\sim 0 \Rightarrow$

$$\frac{\partial}{\partial T}\left(\frac{2\pi m g (kT)^2}{h^2}\left[\frac{1}{2}\frac{\mu^2}{(kT)^2}+\frac{\pi^2}{6}\right]\right) = \frac{\partial}{\partial T}\left(\frac{2\pi m g}{h^2}\frac{\mu^2}{2} + \frac{2\pi m g (kT)^2\pi^2}{6h^2}\right)$$

$$= \frac{4\pi^3 m g\, k^2 T}{6h^2}$$

$$C_V = A\frac{\cancel{4}^2}{\cancel{6}_3 h^2}\pi^3 m g\, k^2 T$$

$$C_V = \frac{2A\pi^3 m g k^2 . T}{3h^2} \Rightarrow \boxed{C_V \propto T}$$

$$C_V = \frac{2\pi A m g}{N h^2}\frac{\pi^2 k^2 T}{3} = \frac{(\pi k)^2 T}{3\varepsilon_F}$$

5.

$$\varepsilon = \frac{p^2}{2m} \mp \overbrace{\mu_B H}^{>0}$$

$-$ spin paralelo: $\vec{s}\uparrow \; \uparrow\vec{H}$ ó $\vec{s}\downarrow \; \vec{H}\downarrow$

$+$ spin antiparalelo: $\vec{s}\downarrow \; \uparrow\vec{H}$ ó $\vec{s}\uparrow \; \vec{H}\downarrow$

Dos posibilidades para spines paralelos entre sí

gas de electrones a temperatura $T=0$

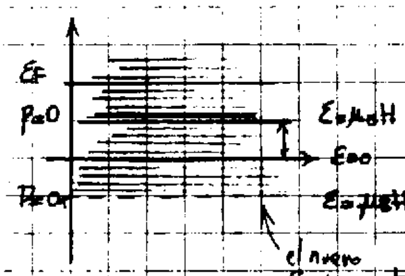
a) Como estamos a temperatura cero, estarán ocupados todos los niveles energéticos hasta la ε_F

← # medio de partículas con spin antiparalelo

$$\langle N\rangle^+ = \frac{V 4\pi g}{h^3}\int_0^{p_0} p^2\, dp \qquad \varepsilon = \frac{p^2}{2m} + \mu_B H$$

$$p = \sqrt{2m(\varepsilon - \mu_B H)}$$

$$= \frac{V 4\pi g}{h^3}\int_{\mu_B H}^{\varepsilon_F} m\,\sqrt{2m(\varepsilon - \mu_B H)}\, d\varepsilon \qquad dp = \frac{2m\, d\varepsilon}{(2m(\varepsilon - \mu_B H))^{1/2}\, 2}$$



$$= \frac{4 V g \sqrt{2}\, m^{3/2}\pi}{h^3}\int_{\mu_B H}^{\varepsilon_F} (\varepsilon - \mu_B H)^{1/2} d\varepsilon \qquad p\, dp = m\, d\varepsilon$$

$$\langle N\rangle^+ = g\frac{4 V 2^{1/2} m^{3/2}\pi}{h^3}\frac{(\varepsilon - \mu_B H)^{3/2}}{3/2}\Bigg|_{\mu_B H}^{\varepsilon_F} \qquad \langle n_i\rangle = \frac{1}{e^{-\beta\mu}e^{\beta\varepsilon}+1}$$

$$\varepsilon_F = \mu \qquad \frac{p_F^2}{2m} \pm \mu_B H = \mu$$

$$\langle N\rangle = \langle N\rangle^+ + \langle N\rangle^-$$

$$\frac{\langle N\rangle^+}{V} = \frac{4 g\, 2^{3/2} m^{3/2}\pi}{3h^3}(\varepsilon_F - \mu_B H)^{3/2}$$

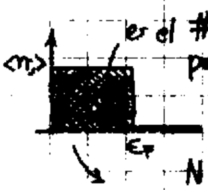
IMPORTANTE

$g=1$ porque el campo ha roto la degeneración

Procediendo en modo idem será:

$$\boxed{\frac{\langle N\rangle^-}{V} = \frac{4\pi (2m)^{3/2}}{3h^3}(\varepsilon_F + \mu_B H)^{3/2}}$$

La densidad será máxima con los spines paralelos al campo porque en dicho caso la energía de Fermi tiene un valor menor, disminuida en $\mu_B H$ y entonces tengo a las partículas más apretadas.



$$N = \int_0^{\varepsilon_F}\langle n_i\rangle\, d\varepsilon$$

$$\left(\frac{3h^3 N_0^{\pm}}{V}\frac{1}{4\pi(2m)^{3/2}}\right)^{2/3} = \varepsilon_F \mp \mu_B H \;\rightarrow\; \varepsilon_F \text{ es menor con } N_0^-$$

b) $\varepsilon_F > \mu_B H \qquad N^+ \rightarrow$ spin antiparalelo, $N^- \rightarrow$ spin paralelo

Cada electrón contribuye a la magnetización con $\mp\mu_B \Rightarrow$

$$M = -\mu_B N^+ + \mu_B N^-$$

$$M = -\mu_B\frac{4\pi V}{3h^3}\left(2m[\varepsilon_F - \mu_B H]\right)^{3/2} + \mu_B\frac{4\pi V}{3h^3}\left(2m[\varepsilon_F + \mu_B H]\right)^{3/2}$$

$$\boxed{M = \mu_B\frac{4\pi V (2m)^{3/2}}{3h^3}\left[-(\varepsilon_F - \mu_B H)^{3/2} + (\varepsilon_F + \mu_B H)^{3/2}\right]}$$

$$\chi_0 = \lim_{H\to 0}\frac{M}{H} \Rightarrow$$

$$\chi_0 = \lim_{H\to 0} \frac{\mu_B 4\pi V (2m)^{3/2}}{3h^3} \cdot \frac{1}{H} \left[-\varepsilon_F^{3/2} \left\{ \left(1 - \frac{\mu_B H}{\varepsilon_F}\right)^{3/2} - \left(1 + \frac{\mu_B H}{\varepsilon_F}\right)^{3/2} \right\} \right]$$

$$\left[\frac{-\varepsilon_F^{\cancel{3/2}\,1/2}}{\cancel{H}} \left(-\cancel{\frac{1}{2}} - \frac{3}{2}\frac{\mu_B \cancel{H}}{\cancel{\varepsilon_F}} + \cancel{\frac{1}{2}} - \frac{3}{2}\frac{\mu_B \cancel{H}}{\cancel{\varepsilon_F}} \right) \right]$$

$$\chi_0 = \frac{\mu_B 4\pi V (2m)^{3/2}}{3h^3} \left(+ \varepsilon_F^{1/2} 3\mu_B \right)$$

$$\boxed{\chi_0 = \frac{4\pi V \mu_B^2 (2m)^{3/2} \varepsilon_F^{1/2}}{h^3}}$$

6.

Gas de electrones en el límite ultrarelativista. Problema 3D

a)

$$N = \frac{4\pi V}{h^3} g \int_0^{p_F} p^2 \, dp \qquad \text{con} \qquad \epsilon = c.p \qquad d\epsilon = c.dp$$

$$N = \sum_\nu 1$$

$$N = \frac{4\pi V g}{h^3} \int_0^{cp_F} \frac{\varepsilon^2}{c^2} \cdot \frac{1}{c} \, d\varepsilon$$

$$N = \frac{V}{h^3} \int d^3p = \frac{g V 4\pi}{h^3} \int_0^{p_F} p^2 \, dp$$

$$N = \frac{4\pi V}{h^3} g \left. \frac{\varepsilon^3}{3} \cdot \frac{1}{c^3} \right|_0^{\varepsilon_F}$$

$$N = \frac{4\pi V}{3h^3} g \, p_F^3 \longrightarrow \frac{\varepsilon_F^3}{c^3} = \left(\frac{3h^3 N}{4\pi V g} \right)$$

$$E = \sum_\nu 1 . \varepsilon_\nu$$

$$\int cp . d^3p$$

$$E = \frac{V}{h^3} g \int \varepsilon \, d^3p = \frac{Vg}{h^3} \iiint \frac{p^3}{c} \operatorname{sen}\theta \, d\theta . d\varphi . dp$$

$$E = \frac{4\pi V}{h^3} g \int_0^{p_F} \frac{p^3}{c} . dp$$

$$\boxed{E = \frac{3}{4} N \varepsilon_F}$$

$$E = \frac{4\pi V}{h^3 4c} g . p_F^4 = \frac{\pi V}{h^3} g \frac{p_F^4}{c}$$

$$N = \frac{4}{3} \frac{E}{p_F c} \longrightarrow \boxed{N = \frac{E}{c} \cdot \frac{4}{3 p_F}}$$

b)

$$E = (3/2) P.V \rightarrow \frac{N c \cancel{3} p_F}{\cancel{4}_{\,2}} = \frac{\cancel{3}}{\cancel{2}} P.V \rightarrow PV = \frac{N}{2} c . p_F = \frac{N}{2} \varepsilon_F$$

Vale en general

$$N = \frac{4\pi V}{h^3} g \int_0^\infty \frac{p^2 . dp}{z^{-1} e^{\beta\varepsilon} + 1} \qquad p^2 . dp = \frac{\varepsilon^2}{c^3} . d\varepsilon \qquad e^{\frac{\mu}{kT}} \ll 1$$

$$\frac{\mu}{kT} \ll 0 \qquad \mu \ll 0$$

$$N = \frac{4\pi V}{h^3} \frac{g}{c^3} \int_0^\infty \frac{\varepsilon^2 . d\varepsilon}{z^{-1} e^{\beta\varepsilon} + 1}$$

$\Rightarrow$ a orden más bajo es $\mu(T=0) = \varepsilon_F$

$T \sim 0$ } Temp. baja

$\frac{\lambda^3}{v} \gg 1$ } densidad alta

$\beta\varepsilon = x$, $d\varepsilon = \frac{dx}{\beta}$

$$N = \frac{4\pi V}{h^3} \frac{g}{c^3 \beta^3} \int_0^\infty \frac{x^{\cancel{2}\,\nu-1} . dx}{z^{-1} e^x + 1}$$

$$N = \frac{4\pi V}{h^3}\frac{g}{c^3\beta^3}\Gamma(3) f_3(z)$$

$$N = \frac{4\pi V}{h^3} g \frac{1}{c^3}(kT)^3 \cancel{2} f_3(z) \qquad f_3(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^3}(-1)^{l+1}$$

$$\frac{Nh^3c^3}{4\pi V g} = (kT)^3 \cdot \int_0^\infty \frac{x^2 dx}{e^{x-\beta\mu}+1}$$ con $T\sim 0$ Lema Sommerfeld

$$\frac{\epsilon_F^3}{3} = (kT)^3 \cdot \left[\int_0^{\beta\mu} x^2 . dx + \frac{\pi^2}{6} 2x \Big|_{\beta\mu} \right]$$

$$\left(\frac{x^3}{3}\Big|_0^{\beta\mu} + \frac{\pi^2}{3}\beta\mu \right)$$

$$\frac{\epsilon_F^3}{\cancel{3}} = (kT)^3 \left(\frac{(\beta\mu)^3}{\cancel{3}} + \frac{\pi^2}{\cancel{3}}\beta\mu \right)$$

$$\boxed{\mu^3 + \pi^2 k^2 T^2 \mu = \epsilon_F^3}$$

$\mu(T)$ al menor orden no nulo en T

Parece funcionar porque en $T=0$ es lo esperado $\mu = \epsilon_F$

c) Gas de electrones en el límite ultrarelativista. Problema 2D

c) a)

$$\langle N \rangle = \sum_p \langle n_p \rangle \longrightarrow \frac{A}{h^2}\int d^2p = \frac{A}{h^2}\iint p \, d\theta . dp = \frac{2\pi A}{h^2}\int_0^\infty p . dp$$

$\varepsilon = cp$

$\frac{1}{c} d\varepsilon = dp$

$$\langle N \rangle = \frac{2\pi A}{h^2}\frac{1}{c^2}\int_0^\infty \varepsilon . d\varepsilon \langle n_\varepsilon \rangle$$

$$\langle N \rangle = \frac{2\pi A}{h^2c^2}\int_0^\infty \frac{\varepsilon \, d\varepsilon}{z^{-1}e^{\beta\varepsilon}+1}$$

$\beta\varepsilon = x$

$\varepsilon = \frac{x}{\beta}$

$d\varepsilon = dx \frac{1}{\beta}$

$$= \frac{2\pi A}{h^2c^2\beta^2}\int_0^\infty \frac{x . dx}{z^{-1}e^x+1}$$

$$\langle N \rangle = \frac{2\pi A}{h^2c^2\beta^2}\underbrace{\Gamma(2)}_{=1} f_2(z)$$ a $T=0$ $\langle n_\varepsilon \rangle = 1$ entre $(0, \epsilon_F)$

$$\langle N \rangle = \frac{2\pi A}{h^2c^2}\int_0^{\epsilon_F} \varepsilon . d\varepsilon = \frac{\cancel{2}\pi A}{h^2c^2}\frac{1}{\cancel{2}}\epsilon_F^2 \rightarrow \boxed{\epsilon_F^2 = \frac{Nh^2c^2}{\pi A}}$$

$$\langle E \rangle = \sum_\varepsilon \langle n_\varepsilon \rangle \varepsilon = \sum_p \langle n_p \rangle cp \rightarrow \sum_p cp \xrightarrow{\text{continuo}} \frac{A}{h^2}\int d^2p . cp$$

$$\langle E \rangle = \frac{Ac}{h^2}\int p . d^2p = \frac{Ac2\pi}{h^2}\int_0^\infty p^2 . dp = \frac{2\pi A \cancel{c}}{h^2}\frac{1}{c^{\cancel{3}2}}\int_0 \varepsilon^2 d\varepsilon \langle n_\varepsilon \rangle$$

$$\langle E \rangle = \frac{2\pi A}{h^2c^2}\int_0^{\epsilon_F} \varepsilon^2 . d\varepsilon = \frac{2\pi A}{h^2c^2}\frac{\epsilon_F^3}{3}$$

$$\left(\frac{E\, h^2 c^2 3}{2\pi A}\right)^{2/3} = \frac{N h^2 c^2}{\pi A} \quad \rightarrow \quad E = \frac{N h^2 c^2}{\pi A} \frac{2^{2/3} \pi^{2/3} A^{2/3}}{h^{4/3} c^{4/3} 3^{2/3}} = \boxed{N \frac{(hc)^{2/3}}{(\pi A)^{1/3}} \left(\frac{4}{9}\right)^{1/3} = E}$$

c) b)

$$\langle N \rangle = \frac{2\pi A}{h^2 c^2 \beta^2} f_2(z)$$

$$\cdot \int_0^\infty \frac{x\, dx}{\underbrace{z^{-1} . e^x}_{e^{x-\beta\mu}} + 1}$$

$$\langle N \rangle \sim \frac{2\pi A}{(hc\beta)^2} \left[\int_0^{\beta\mu} x\, dx + \frac{\pi^2}{6} \right] = \frac{2\pi A}{(hc\beta)^2} \left(\frac{x^2}{2} \Big|_0^{\beta\mu} + \frac{\pi^2}{6} \right)$$

$$\langle N \rangle = \frac{\cancel{2}\pi A}{(hc\beta)^2} \left(\frac{(\beta\mu)^2}{\cancel{2}_1} + \frac{\pi^2}{\cancel{6}_3} \right)$$

$$\frac{N h^2 c^2}{\pi A} = \left(\mu^2 + \frac{\pi^2}{\beta^2 3} \right)$$

$$\epsilon_F^2 = \mu^2 + \frac{\pi^2}{3} k^2 T^2$$

$$\boxed{\mu = \left[\epsilon_F^2 - \left(\frac{\pi}{3} k T\right)^2 \right]^{1/2}}$$