Serie 2: Procesos estocásticos. Cadenas de Markov

1.

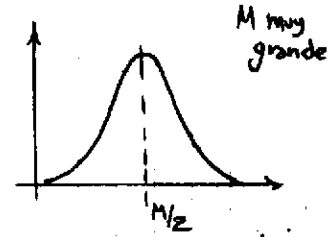
$$X_i \begin{cases} 1 & \text{con prob. } p \\ 0 & \text{con prob } 1-p \end{cases}$$

son independientes los X_i

$$N = \sum_{i=1}^{M} X_i$$, N variable aleatoria. N será el # de "1" obtenidos

Será una distribución binomial:

$$P(N) = \frac{M!}{N!\,(M-N)!}\, p^N (1-p)^{M-N}$$



Variables aleatorias de este tipo serían:

* $\sum$ # caras y cecas de arrojar una moneda N veces

2.

$$\text{Lím } M \to \infty,\ p \to 0,\ M.p = \lambda$$

$$p_n = \binom{M}{n} p^n (1-p)^{M-n} \qquad \binom{M}{n} = C_M^n = \frac{M!}{n!\,(M-n)!}$$

Aproximación de Stirling

$$\boxed{\ln M! = M \ln M - M \quad \text{si } M \gg 1}$$

$$\ln\left(\frac{M!}{n!(M-n)!}\right) = \frac{M}{n(M-n)} \cdot \ln\left(\frac{M}{n(M-n)}\right) - \frac{M}{n(M-n)}$$

$$\ln\left(\frac{M!}{n!\,(M-n)!}\right) = \ln M! - \ln[n!\,(M-n)!]$$

Uso Stirling

$$\ln\left(\frac{M!}{n!(M-n)!}\right) = \ln M! - \ln n! - \ln (M-n)!$$

$$= M.\ln M - M - n.\ln(n) + n - (M-n).\ln(M-n) + (M-n)$$

$$= M.\ln M \cancel{-M} - n.\ln(n) \cancel{+n} - M.\ln(M-n) + n.\ln(M-n) \cancel{+M} \cancel{-n}$$

$$= (n-M).\ln(M-n) + M.\ln M - n.\ln(n)$$

n es finito

$$\lim_{\substack{M\to\infty \\ p\to 0 \\ Mp \to \lambda}} (1-p)^{M-n} = \lim_{M\to\infty}\left(1 - \frac{\lambda}{M}\right)^{M-n}$$

$$= \lim_{M\to\infty}\left(1-\frac{\lambda}{M}\right)^M \underbrace{\left(1-\frac{\lambda}{M}\right)^{-n}}_{=1}$$

$$= e^{-\lambda}$$

Puede evaluarse el límite directamente sin utilizar la aproximación del logaritmo:

$$p_n = \left(\frac{M!}{n!(M-n)!}\right) p^n \cdot (1-p)^{M-n}$$

$$\lim_{\substack{M\to\infty \\ p\to 0 \\ MP\to\lambda}} p_n = \lim_{\substack{M\to\infty \\ p\to 0 \\ MP\to\lambda}} \frac{M!}{n!(M-n)!}\, p^n (1-p)^{M-n} = \lim_{M\to\infty} \frac{M!}{n!(M-n)!} \frac{\lambda^n}{M^n}\left(1 - \frac{\lambda}{M}\right)^{M-n}$$

$$\lim_{M\to\infty} \frac{\overbrace{M(M-1)\ldots(2)(1)}^{M \text{ factores}}\ \lambda^n \left(1 - \lambda/M\right)^M \left(1-\lambda/M\right)^{-n}}{\underbrace{n!}_{n \text{ factores}}\ \underbrace{(M-n)(M-n-1)\ldots(M-n-2)\ldots(2)(1)}_{M-n \text{ factores}} \cdot M^n}$$

$$\lim_{M\to\infty} \frac{\overbrace{M(M-1)(M-2)\dots(M-n+1)}^{n \text{ factores}}\cancel{(M-n)!}}{n!\,\cancel{(M-n)!}} \frac{\lambda^n}{M^n}\left(1+\frac{\lambda}{M}\right)^M\left(1+\frac{\lambda}{M}\right)^{-n}$$

$$\lim_{M\to\infty} 1\left(1-\frac{1}{M}\right)\left(1-\frac{2}{M}\right)\cdots\left(1-\frac{n}{M}+\frac{1}{M}\right)\cdot\frac{1}{n!}\lambda^n\left(1+\lambda/M\right)^M\left(1+\lambda/M\right)^{-n}$$

$$\boxed{\lim_{M\to\infty} P_n = \frac{\lambda^n}{n!}e^{-\lambda}}$$

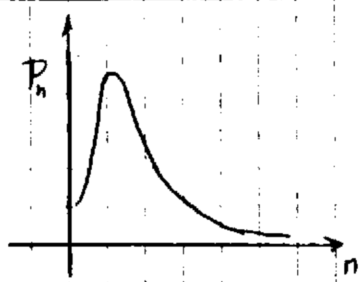
$\Rightarrow$ El límite de la distribución binómica cuando $M\to\infty$, $p\to 0$, $MP\to\lambda$ es la distribución de Poisson.

$M.P \equiv \lambda$

Corresponde a variables que presentan una cantidad enorme de sucesos, c/u de los cuales tiene una probabilidad muy pequeña.

① Las moléculas de un gas ideal encerrado en un contenedor podrían ser un ejemplo: Podemos pensar en celdas ocupadas y vacías. $\begin{cases} n & \text{ocupadas} \\ N-n & \text{vacías}\end{cases}$

② Un sistema de N spines en un campo H es un ejemplo que sigue la distrib. de Poisson



3■.

$$\bar{n} = \sum_{n=1}^{N} P_n . n$$

* **distribución de Poisson**

$$\bar{n} = \sum_{n=1}^{M} \frac{\lambda^n}{n!} e^{-\lambda}. n \quad \text{, con } M\to\infty$$

$k = n-1$

$$= e^{-\lambda} \sum_{k=0}^{M-1} \frac{\lambda^{k+1}}{(k+1)!}(k+1)$$

$$= e^{-\lambda}.\lambda \sum_{k=0}^{M-1} \frac{\lambda^k \cancel{(k+1)}}{\cancel{(k+1)}\,k!} = e^{-\lambda}.\lambda\sum_{k=0}^{M-1}\frac{\lambda^k}{k!}$$

$$\Rightarrow \text{si } M\to\infty \Rightarrow \quad \bar{n} = e^{-\lambda}.\lambda.e^{\lambda} \to$$

$$\boxed{\bar{n}=\lambda}$$

$$\overline{(\Delta n)^2} = \overline{(n-\bar{n})^2} \to$$
$$= \overline{n^2 - 2n\bar{n} + \bar{n}^2}$$
$$\overline{n^2} - \underbrace{\overline{2n\bar{n}}}_{2\bar{n}^2} + \bar{n}^2$$
$$\overline{(\Delta n)^2} = \overline{n^2} - \bar{n}^2$$

$$\overline{(\Delta n)^2} = \sum_{n=1}^{M} \frac{\lambda^n}{n!}e^{-\lambda}(n-\bar{n})^2 \quad ; \text{ pero:}$$

$$\overline{n^2} = \sum_{n=1}^{M} \frac{\lambda^n}{n!}e^{-\lambda}(n^2) = e^{-\lambda}\sum_{n=1}^{M}\frac{\lambda^n n}{(n-1)!}$$

$$\overline{n^2} = e^{-\lambda}\left(\sum_{n=1}^{M}\frac{\lambda^n(n-1)}{(n-1)!} + \sum_{n=1}^{M}\frac{\lambda^n . 1}{(n-1)!}\right)$$

$$\overline{n^2} = e^{-\lambda}\left(\sum_{n=2}^{M}\frac{\lambda^n}{(n-2)!} + \sum_{n=1}^{M}\frac{\lambda^n}{(n-1)!}\right)$$

$$e^{-\lambda}\left(\sum_{k=0}^{M}\frac{\lambda^k.\lambda^2}{k!} + \sum_{k=0}^{M}\frac{\lambda^k.\lambda}{k!}\right)$$

$$e^{-\lambda}\left(\lambda^2.e^{\lambda} + \lambda.e^{\lambda}\right) = \lambda^2+\lambda$$

$$\overline{(\Delta n)^2} = \overline{n^2} - \bar{n}^2 = \lambda^2+\lambda-\lambda^2 = \lambda$$

$$\boxed{\overline{(\Delta n)^2} = \lambda}$$

* <u>Distribución binomial</u>

$$\bar n = \sum_{n=1}^{N} \frac{N!\, n}{n!\,(N-n)!}\, p^n (1-p)^{N-n} = \sum_{n=1}^{N} \frac{N!}{(n-1)!\,(N-n)!}\, p^n (1-p)^{N-n}$$

Usando teorema del binomio se tiene:

$$[p+(1-p)]^N = 1^N = 1 = \sum_{n=0}^{N} \frac{N!}{(N-n)!}\,\frac{p^n(1-p)^{N-n}}{n!} \Rightarrow$$

tomamos $k=n-1$ →

$$= \sum_{k=0}^{N-1} \frac{N!}{k!\,(N-k-1)!}\, p^{k+1}(1-p)^{N-k-1}$$

tomamos $M=N-1$

$$= \sum_{k=0}^{M} \frac{N.M!}{k!\,(M-k)!}\, p.p^k(1-p)^{M-k}$$

$$= N.p \sum_{k=0}^{n} \frac{M!}{(M-k)!}\,\frac{p^k(1-p)^{M-k}}{k!} \Rightarrow \boxed{\bar n = N.p}$$

$$\overline{n^2} = \sum_{n=1}^{N} \frac{N!\, n}{(n-1)!\,(N-n)!}\, p^n (1-p)^{N-n} = N.p \sum_{n=1}^{N} \frac{(N-1)!\, n}{(n-1)!\,(N-n)!}\, p^{n-1}\cdot(1-p)^{N-n}$$

$$= N.p\left(\sum_{k=0}^{N-1} \frac{(N-1)!\,(k+1)}{(N-1-k)!}\,\frac{p^k}{k!}(1-p)^{N-1-k}\right) \qquad M=N-1$$

$$= N.p.\left[\sum_{k=0}^{M} \frac{M!}{(M-k)!}\,\frac{p^k}{k!}(1-p)^{M-k}\,(1+k)\right]$$

Estoy sumando de $k=0$ a $k=M$ ⇒ tengo $M+1$ términos, es decir $M+1=N$ términos. Pero el teorema del binomio tiene $N+1$ términos pues la Σ va desde $n=0$ a $n=N$ ⇒

$$1 = \sum_{n=0}^{N-1} \frac{N!}{(N-n)!}\,\frac{p^n(1-p)^{N-n}}{n!} + \overbrace{\frac{\cancel{N!}}{0!}\,\frac{p^N(1-p)^0}{\cancel{N!}}}^{p^N}$$

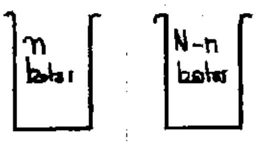
$$= \sum_{n=1}^{N} \frac{N!\,n.p^n(1-p)^{N-n}}{(n-1)!\,(N-n)!} = \sum_{n=1}^{N} \frac{N!\,(n-1)\,p^n(1-p)^{N-n}}{(n-1)!\,(N-n)!} + \sum_{n=1}^{N} \frac{N!\,p^n(1-p)^{N-n}}{(n-1)!\,(N-n)!}$$

$$= \sum_{k=0}^{N-1} \frac{N!\,k\,p^{k+1}(1-p)^{N-1-k}}{k!\,(N-1-k)!} + \sum_{k=0}^{N-1} \frac{(N-1)!\,N.p\,p^k(1-p)^{N-1-k}}{k!\,(N-1-k)!}$$

tomo $N-1=M$ $k=n-1$

$$NP\sum_{k=0}^{N-1} \frac{(N-1)!\,p^k(1-p)^{N-1-k}\,k}{k!\,(N-1-k)!}$$

4.



N bolas

Cada segundo se elige al azar una bola y se la transfiere de urna. n es el # de bolas en una de las urnas.

$$T_{nn'} = \frac{n'}{N}\delta_{n+1,n'} + \left(1-\frac{n'}{N}\right)\delta_{n-1,n'}$$

Esta matriz no tiene elementos diagonales y son $\neq 0$ sólo los laterales inmediatos de la misma

$$\begin{array}{c|ccccc}
 & & n-1 & n & n+1 & \\
\hline
n-1 & \cdots & 0 & \frac{n}{N} & 0 & \cdots \\
n & \cdots & \left(1-\frac{n-1}{N}\right) & 0 & \left(\frac{n+1}{N}\right) & \cdots \\
n+1 & \cdots & 0 & \left(1-\frac{n}{N}\right) & 0 & \cdots
\end{array}$$

5.

Un estado transitorio se define como aquel que no es absorbente. Una cadena de Markov que tenga un estado absorbente y uno transitorio tiene como matriz de transición:

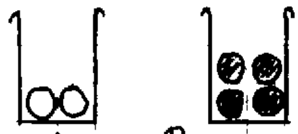
$$Q = \left(\begin{array}{c|c} T & R \\ \hline 0 & \mathbb{1} \end{array}\right)$$, donde R en general son $\neq 0$

A largo t el estado absorbente termina comiéndose al transitorio porque si $R \neq 0$ significa que hay probabilidad no nula de que a cada step caiga algo en el estado absorbente (del cual nunca sale) $\rightarrow$ $n \to \infty$ $Q^n = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline 0 & \mathbb{1} \end{array}\right)$

Si de entrada tenemos $R = 0$ es como dos problemas separados: no hay conexión T evolucionará hacia su equilibrio.

6.

Inicial →



A cada paso se selecciona al azar una bola de cada urna y se intercambian.

No varía el # de bolas en las urnas ; pero sí variará la proporción de colores. En la urna A tendremos:

bb br rr

a)

Pr tomar de B	Pr tomar de A	en A	
$\frac{1}{4} \cdot \frac{1}{2}$	=	(BR)	= $\frac{1}{8}$
$\frac{3}{4} \cdot \frac{1}{2}$	=	RR	= $\frac{3}{8}$
$\frac{1}{4} \cdot \frac{1}{2}$	=	BB	= $\frac{1}{8}$
$\frac{3}{4} \cdot \frac{1}{2}$	=	BR	= $\frac{3}{8}$ → $\frac{1}{2}$

f)

i)

$$\begin{array}{c} \\ BB \\ BR \\ RR \end{array} \begin{array}{c} \begin{array}{ccc} BB & BR & RR \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/8 & 1/2 & 3/8 \\ 0 & 1/2 & 1/2 \end{pmatrix} \end{array}$$

$$\Rightarrow \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/8 & 1/2 & 3/8 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

* Autovalores :

$$|Q - \lambda \mathbb{I}| = 0 \rightarrow$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1/8 & 1/2-\lambda & 3/8 \\ 0 & 1/2 & 1/2-\lambda \end{vmatrix} = -\lambda(1/2-\lambda)^2 - (1/2-\lambda)1/8 + 3/8 \, 1/2 \, \lambda = 0$$

$$-\lambda^3 + \lambda^2 - 1/4\lambda - 1/16 + \lambda/8 + 3/16\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 1/16\lambda - 1/16 = 0$$

$$-\lambda(\lambda^2 - 1/16) + \lambda^2 - 1/16 = 0$$

$$(\lambda^2 - 1/16)(1-\lambda) = 0$$

$\lambda = 1$
$\lambda = 1/4$
$\lambda = -1/4$

$$\boxed{\lambda = \begin{cases} 1 \\ 1/4 \\ -1/4 \end{cases} \quad \Psi = \begin{cases} (1,1,1) \\ (4,1,-2) \\ (-4,1,-2/3) \end{cases}}$$

autovalores y autovectores a derecha

* autovectores a derecha

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 1/8 & 1/2-\lambda & 3/8 \\ 0 & 1/2 & 1/2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-a + b = 0 \rightarrow b = a$

$1/8 a - b/2 + 3/8 c = 0$

$\Psi_1 = (1,1,1)$

$\frac{3}{8} c = 3/8 a \rightarrow a = c$

$-1/4 a + b = 0$

$b = \frac{a}{4}$

$4b$
b
$-2b$
$\rightarrow \Psi_2 = (4,1,-2)$

$b/2 + c/4 = 0 \quad c = -b2$

$a/4 + b = 0$

$b = -a/4 \quad b/2 = -3/4 c$

$-4b$
b
$-2/3 b$
$\rightarrow \Psi_3 = (-4,1,-2/3)$

* Autovectores a izquierda

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1/8 & 1/2-\lambda & 3/8 \\ 0 & 1/2 & 1/2-\lambda \end{vmatrix} = -\lambda(1/2-\lambda)^2 + \frac{3}{8}\cdot\frac{1}{2}\lambda - (1/2-\lambda)1/8 = 0$$

$$-\frac{\lambda}{4} + \lambda^2 - \lambda^3 + \frac{3}{16}\lambda - \frac{1}{16} + \frac{\lambda}{8} = 0$$

$$-\lambda^3 + \lambda^2 + \lambda/16 - 1/16 = 0$$

$$-\lambda(\lambda^2 - 1/16) + \lambda^2 - 1/16$$

$$(\lambda^2 - 1/16)(1-\lambda) = 0$$

Los autovalores serán los mismos:

$$\lambda_1 = 1$$
$$\lambda_2 = 1/4$$
$$\lambda_3 = -1/4$$

$$(a \;\; b \;\; c)\begin{pmatrix} -\lambda & 1 & 0 \\ 1/8 & 1/2-\lambda & 3/8 \\ 0 & 1/2 & 1/2-\lambda \end{pmatrix} = (0 \;\; 0 \;\; 0)$$

$$\begin{cases} -a + \frac{b}{8} = 0 \\ \frac{3}{8}b - \frac{c}{2} = 0 \end{cases} \quad a = b/8 \quad c = \frac{3}{4}b \qquad \chi_1 = (b/8, b, 3/4 b) \qquad 1 = b(1/8 + 1 + 3/4) = b\frac{15}{8}$$

$$\begin{cases} -\frac{a}{4} + \frac{b}{8} = 0 \\ \frac{3}{8}b + \frac{c}{4} = 0 \end{cases} \quad a = \frac{1}{2}b \quad c = -\frac{3}{2}b \qquad \chi_2 = \left(\frac{b}{2}, b, -\frac{3}{2}b\right)$$

normalización

$$\frac{b}{2} + b + \frac{3}{2}b \cdot 8 = 1 \qquad b = 1/6$$

$$\begin{cases} \frac{a}{4} + \frac{b}{8} = 0 \\ \frac{3}{8}b + \frac{3}{4}c = 0 \end{cases} \quad a = -\frac{b}{2} \quad c = -\frac{1}{2}b \qquad \chi_3 = (-b/2, b, -1/2 b)$$

normalización

$$\frac{b}{2} + b + \frac{b}{2}\cdot\frac{2}{3} = 1 \qquad b = \frac{3}{10}$$

$$\boxed{\begin{aligned} \chi_1 &= (1/15,\ 8/15,\ 6/15) \\ \chi_2 &= (1/12 \;\; 1/6 \;\; -1/4) \\ \chi_3 &= (-3/20 \;\; 3/10 \;\; -3/20) \end{aligned}}$$

b)

$$Q^3 = 1^3\begin{pmatrix}1\\1\\1\end{pmatrix}(1/15 \;\; 8/15 \;\; 2/5) + (1/4)^3\begin{pmatrix}4\\1\\-2\end{pmatrix}(1/12 \;\; 1/6 \;\; -1/4) + (-1/4)^3\begin{pmatrix}-4\\1\\-2/3\end{pmatrix}(-3/20 \;\; 3/10 \;\; -3/20)$$

$$Q^3 = \begin{pmatrix} 1/15 & 8/15 & 2/5 \\ 1/15 & 8/15 & 2/5 \\ 1/15 & 8/15 & 2/5 \end{pmatrix} + 1/4^3\begin{pmatrix} 1/3 & 2/3 & -1 \\ 1/12 & 1/6 & -1/4 \\ -1/6 & -1/3 & 1/2 \end{pmatrix} - 1/4^3\begin{pmatrix} +3/5 & -6/5 & 3/5 \\ -3/20 & 3/10 & -3/20 \\ 1/10 & -1/5 & 1/10 \end{pmatrix}$$

vía calculadora

$$Q^3 = \begin{pmatrix} 0{,}0625 & 0{,}5625 & 0{,}375 \\ 0{,}0703125 & 0{,}53125 & 0{,}3984375 \\ 0{,}0625 & 0{,}53125 & 0{,}40625 \end{pmatrix}$$

$$Q^3 = \begin{pmatrix} 1/16 & 9/16 & 6/16 \\ 9/128 & 17/32 & 51/128 \\ 1/16 & 17/32 & 13/32 \end{pmatrix}$$

$$Q^3 = \begin{pmatrix} [1/15 + 1/4^3\cdot 3 - 3/4^3 5] \\ [1/15 + \frac{1}{4^3 12} + \frac{3}{4^3 20}] \end{pmatrix}$$

$$Q^3 = \begin{pmatrix} 1/16 \\ 9/128 \\ \end{pmatrix}$$

se puede ver que da lo esperado

Para tres pasos:

$$\vec{P}(3) = \vec{P}(0) . \overleftrightarrow{Q}^3$$, donde $\vec{P}(0)$ es el vector de la distribución inicial (el tacho de con 2 bolas blancas)

$$(P_{BB}\ P_{BR}\ P_{RR}) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/16 & 9/16 & 6/16 \\ 9/128 & 17/32 & 51/128 \\ 1/16 & 17/32 & 13/32 \end{pmatrix}$$

$$\boxed{\vec{P}(3) = (P_{BB}\ P_{BR}\ P_{RR}) = \begin{pmatrix} 1/16 & 9/16 & 6/16 \end{pmatrix}}$$

← Luego de 3 pasos ya tenemos prob. apreciable de tener la configuración RR

Para n pasos tendremos:

$$\vec{P}(n) = \vec{P}(0) . \overleftrightarrow{Q}^n$$, con $$\overleftrightarrow{Q}^n = \begin{pmatrix} [\chi_1] \\ [\chi_1] \\ [\chi_1] \end{pmatrix}$$

$$(P_{BB}\ P_{BR}\ P_{RR}) = (1, 0, 0) \begin{pmatrix} 1/15 & 8/15 & 2/5 \\ 1/15 & 8/15 & 2/5 \\ 1/15 & 8/15 & 2/5 \end{pmatrix}$$

$$\boxed{\vec{P}(n) = (1/15, 8/15, 2/5)}$$

← Luego de n pasos se ha llegado a la distribución de probabilidad estacionaria

c)

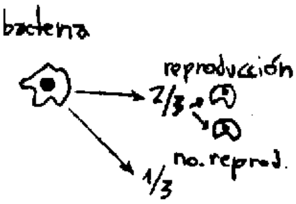
$$\vec{P}(5) = \vec{P}(0) . \overleftrightarrow{Q}^5$$

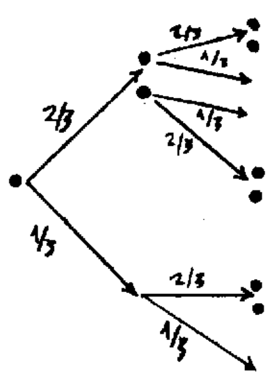
$$= \vec{P}(0) . \left[\lambda_1^5\, \psi_1 . \chi_1 + \lambda_2^5\, \psi_2 . \chi_2 + \lambda_3^5\, \psi_3 . \chi_3 \right]$$

$$\vec{P}(5) = (1, 0, 0) \left[1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1/15 & 8/15 & 2/5 \end{pmatrix} + \left(\tfrac{1}{4}\right)^5 \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1/12 & 1/6 & -1/4 \end{pmatrix} + \left(-\tfrac{1}{4}\right)^5 \begin{pmatrix} -4 \\ 1 \\ -2/3 \end{pmatrix} \begin{pmatrix} -3/20 & 3/10 & -3/20 \end{pmatrix} \right]$$

7.

bacteria





Los pasos serán segundos de tiempo.

si # bacterias es $n \geq 4$ → mueren todas salvo 1

No se llega a tener $n = 4$ ⇒

a)

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1/3 & 2/3 & 0 \\ 4/9 & 1/9 & 4/9 \\ 26/27 & 0 & 1/27 \end{pmatrix} \end{array}$$

← si se reproducen las dos $n=4$ → vuelvo a $n=1$

Con 3 bacterias (las bacterias son distinguibles).

2 NR 4/9, 1 R 2/3 → n=1 $\quad \frac{2}{27} 3 \to \frac{3!}{2!}$

2 R 4/9, 1 NR 1/3 → n=4 $\quad \frac{1}{27} . 3$

3 R 8/27 → n=1

$$+ \frac{6}{27} + \frac{12}{27} + \frac{8}{27}$$

la matriz de transición será:

$$\boxed{Q = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 4/9 & 1/9 & 4/9 \\ 26/27 & 0 & 1/27 \end{pmatrix}}$$

b) El estado estacionario es el autovector a izquierda $\vec{\pi}_1$ de autovalor $\lambda=1$

$$\begin{pmatrix} 1/3-1 & 2/3 & 0 \\ 4/9 & 1/9-1 & 4/9 \\ 26/27 & & 1/27-1 \end{pmatrix} = \begin{pmatrix} -2/3 & 2/3 & 0 \\ 4/9 & -8/9 & 4/9 \\ 26/27 & 0 & -26/27 \end{pmatrix} \rightarrow$$

$$(a\ b\ c)\begin{pmatrix} -2/3 & 2/3 & 0 \\ 4/9 & -8/9 & 4/9 \\ 26/27 & 0 & -26/27 \end{pmatrix} = (0\ 0\ 0)$$

$$-\frac{2}{3}a + \frac{4}{9}b + \frac{26c}{27} = 0 \rightarrow a = \frac{3}{2}\left(+\frac{26}{27}\cdot\frac{2}{13}b + \frac{4}{9}b\right) = \frac{4}{3}b$$

$$\frac{4}{9}b - \frac{26}{27}c = 0 \rightarrow c = \frac{4}{9}\cdot\frac{27}{26}b$$

$$\vec{\pi}_1 = \left(\frac{4}{3}b,\ b,\ \frac{6}{13}b\right)$$

$$\left(\frac{4}{3}+1+\frac{6}{13}\right)b = 1$$

$$b = \frac{39}{109}$$

$$\vec{\pi}_1 = \left(\frac{4}{3}\frac{39}{109},\ \frac{39}{109},\ \frac{6}{13}\frac{39}{109}\right)$$

$$\boxed{\vec{\pi}_1 = \left(\frac{52}{109},\ \frac{39}{109},\ \frac{18}{109}\right)}$$

c)

$$\vec{P}(2) = \vec{P}(0)\ \overleftrightarrow{Q}^2$$

$\vec{P}(0)$ corresponde al estado inicial de tener $n=2$ bacterias con prob. 1

$$= (0\ \ 1\ \ 0)\begin{pmatrix} 1/3 & 2/3 & 0 \\ 4/9 & 1/9 & 4/9 \\ 26/27 & 0 & 1/27 \end{pmatrix}^2$$

con calculadora

$$= (0\ \ 1\ \ 0)\begin{pmatrix} 11/27 & 8/27 & 8/27 \\ 152/243 & 25/81 & 16/243 \\ 260/729 & 52/81 & 1/729 \end{pmatrix}$$

$$\boxed{\vec{P}(2) = \left(\frac{152}{243},\ \frac{25}{81},\ \frac{16}{243}\right)}$$

Vemos que tiene la normalización correcta y pinta de $\vec{\pi}_1$

d) Si las bacterias tienen una cierta probabilidad de morir, entonces existe un estado $n=0$ y será

$$\overleftrightarrow{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{pmatrix} \end{array}$$

, donde el estado 0 es absorbente porque si No hay bacterias no se pueden generar de la nada.

Ahora la matriz Q, si bien es estocástica no será regular porque $\nexists\ n: \ (\overleftrightarrow{Q}^n)_{ij} > 0 \ \ \forall i,j$, lo cual puede verse ensayando el cuadrado de Q

$$Q^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_{21} & q_{22} & \cdots & q_{21} \\ \vdots & & q_{31} & \vdots \\ q_{41} & & & q_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_{21} & \cdots & & q_{21} \\ \vdots & \ddots & & \vdots \\ q_{41} & \cdots & & q_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{pmatrix}$$

Se ve que Q^2 mantiene la fila absorbente y esto es similar para (inductivamente) Q^n

Asimismo puede verse que: $\lim_{n \to \infty} Q^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, con lo cual el estado estacionario es la muerte de todas las bacterias: es decir el estado $\boxed{n=0}$

8.

$$F(z,t) = \sum_n P(n,t) \cdot z^n \quad ; \quad P(n,t) \text{ probabilidad}$$

a)

- $F(1,t) = \sum_n^N P(n,t)\, 1^n$

$$F(1,t) = \sum_n^N P(n,t) = 1$$

Porque estamos sumando sobre todo el espacio muestral. Asimismos correcta normalización de $P(n,t)$.

- $$\frac{\partial}{\partial z} F(z,t)\Big|_{z=1} = \sum_n^N \frac{\partial}{\partial z}(z^n) \cdot P(n,t) \Big]_{z=1}$$

$$= \sum_n^N n\, z^{n-1} P(n,t) \Big]_{z=1}$$

$$= \sum_n^N n \cdot P(n,t)$$

$$\frac{\partial}{\partial z}(F(z,t))\Big|_{z=1} = \langle n \rangle_t$$

- $$\frac{\partial^2}{\partial z^2}(F(z,t))\Big|_{z=1} = \sum_n^N n\, z^{n-2} \cdot (n-1) \cdot P(n,t) \Big]_{z=1}$$

$$= \sum^N n(n-1) P(n,t)$$

$$= \sum_n^N n^2 P(n,t) - \sum_n^N n \cdot P(n,t)$$

$$\frac{\partial^2}{\partial z^2}[F(z,t)]\Big|_{z=1} = \langle n^2 \rangle_t - \langle n \rangle_t$$

b)

$$\dot{P}_n = P_{n+1} + P_{n-1} - 2P_n \quad ; \quad (-\infty < n < +\infty)$$

caminata aleatoria simétrica

$$\frac{\partial P_n}{\partial t} = P_{n+1} + P_{n-1} - 2P_n \longrightarrow \text{le meto una } \sum_n(\;)z^n \text{ en cada miembro}$$

$$\frac{\partial}{\partial t}\left(\sum_n P_n z^n\right) = \sum_n P_{n+1} \cdot z^n + \sum_n P_{n-1} \cdot z^n - 2 \sum_n P_n \cdot z^n$$

$$= \frac{1}{z} \sum_{n'} P_{n'} \underline{z^{n'}} + z \sum_{n'} P_{n'} z^{n'} - 2 \sum_n P_n \cdot z^n$$

$$\frac{\partial}{\partial t}\left(\sum_n P_n Z^n\right) = \left(\frac{1}{Z} + Z - 2\right)\left(\sum_n P_n Z^n\right)$$

$$\frac{\partial}{\partial t} F = \left(\frac{Z^2+1-2Z}{Z}\right) F \quad \rightarrow \quad \left(\frac{1}{F}\right)\partial F = \frac{(Z-1)^2}{Z}\,\partial t$$

$$\ln F = \frac{(Z-1)^2}{Z}\, t$$

$$F = C.e^{\frac{(Z-1)^2}{Z}.t}$$

$$F(Z,t) = \sum_n P_n . Z^n = C e^{\frac{(Z-1)^2}{Z} t}$$

$$F(Z,t) = \sum_{n=-\infty}^{\infty} P_n . Z^n = C. \sum_{\ell=0}^{\infty} \frac{(Z-1)^{2\ell}\, t^\ell}{\ell!\; Z^\ell}$$

* __Contornos__

Pero $F(Z=1,t) = 1 \rightarrow$

$$1 = C. \sum_{k=0}^{\infty} \frac{0^{2k}}{1} \frac{t^k}{k!} = C. \quad \rightarrow \quad C = 1$$

solo vive $k=0$

$$F(Z,t) = e^{\frac{(Z-1)^2}{Z} t} = e^{\left[Z + \frac{1}{Z} - 2\right] t} = e^{Zt} . e^{\frac{t}{Z}}\, e^{-2t}$$

$$F(Z,t) = e^{-2t} \sum_{j=0}^{\infty} \frac{(Zt)^j}{j!} \sum_{k=0}^{\infty} \frac{t^k}{k!\, Z^k}$$

$$\sum_{n=0}^{\infty} P_n(t)\, Z^n = e^{-2t} \sum_{j,k=0}^{\infty} \frac{Z^{j-k}\, t^{j+k}}{j!\, k!} \qquad j-k=n$$

$$\sum_{n=0}^{\infty} Z^n . P_n(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{e^{-2t} . Z^n\, t^{2k+n}}{(n+k)!\; k!} \quad \Rightarrow$$

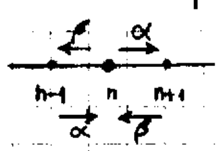
$$\boxed{P_n(t) = \sum_{k=0}^{\infty} \frac{e^{-2t} . t^{2k+n}}{(n+k)!\; k!}}$$

9.

→ prob de moverse a der. → α

→ prob de moverse a izq → β

$$P_n(t) = g_n(t) \left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} e^{-\left[\alpha + \beta - 2\sqrt{\alpha\beta}\right] t}$$



$$\frac{\partial P(n,t)}{\partial t} = \alpha P_{n-1} + \beta P_{n+1} - \alpha P_n - \beta . P_n$$

$$\dot{P}_n(t) = \alpha . P_{n-1} + \beta . P_{n+1} - (\alpha + \beta) P_n$$

$$\left.\begin{aligned} \alpha &= P_{1/1}(n, n+1) \\ \beta &= P_{1/1}(n, n-1) \end{aligned}\right\} \Rightarrow$$

$$\sum_n \dot{P}_n(t) . Z^n = \sum_n \alpha\, P_{n-1} Z^{n-1} . Z + \beta . P_{n+1} \frac{Z^{n+1}}{Z} - (\alpha + \beta) P_n . Z^n$$

$$\boxed{\frac{\partial}{\partial t}\big(F(Z,t)\big) = \left(\alpha Z + \beta \frac{1}{Z} + [\alpha + \beta]\right) F(Z,t)}$$

serán

$$\left.\begin{aligned} \beta &= P_{1/1}(n+1, n) \\ \alpha &= P_{1/1}(n-1, n) \end{aligned}\right\}$$

Esta sería la ecuación para la función generatriz pero es harto complicada; probamos el reemplazo solicitado →

$$\frac{\partial P_n}{\partial t} = \alpha\, g_{n-1} \left(\frac{\beta}{\alpha}\right)^{-\frac{n-1}{2}} e^{-[\quad]t} + \beta\, g_{n+1} \left(\frac{\beta}{\alpha}\right)^{-\frac{n+1}{2}} e^{-[\quad]t} - (\alpha + \beta)\, g_n \left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} . e^{-[\quad]t}$$

$$\frac{\partial P_n}{\partial t} = \frac{\partial q_n}{\partial t}\left(\frac{\beta}{\alpha}\right)^{\frac{n}{2}} e^{-[\ \ ]t} - q_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} e^{-[\ \ ]t}\cdot(\alpha+\beta-2\sqrt{\alpha\beta}) =$$

$$\alpha\, q_{n-1}\left(\text{—}\right)^{-\frac{n-1}{2}} e^{-[\ \ ]t} + \beta\, q_{n+1}\left(\frac{\beta}{\alpha}\right)^{-\frac{n+1}{2}} e^{-[\ \ ]t} - (\alpha+\beta)\, q_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} e^{-[\ \ ]t}$$

$$\dot{q}_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} = q_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}(\alpha+\beta-2\sqrt{\alpha\beta}) + \alpha\, q_{n-1}\left(\frac{\beta}{\alpha}\right)^{-\frac{n-1}{2}} + \beta\, q_{n+1}\left(\frac{\beta}{\alpha}\right)^{-\frac{n+1}{2}} - (\alpha+\beta)\, q_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}$$

$$\dot{q}_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}} = q_n\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}[\cancel{\alpha}+\cancel{\beta}-2\sqrt{\alpha\beta}-\cancel{\alpha}-\cancel{\beta}] + \alpha\, q_{n-1}\left(\frac{\beta}{\alpha}\right)^{\frac{1-n}{2}} + \beta\, q_{n+1}\left(\frac{\beta}{\alpha}\right)^{\frac{-n-1}{2}}$$

$$\dot{q}_n\cancel{\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}} = -q_n\cancel{\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}}2\sqrt{\alpha\beta} + \alpha\, q_{n-1}\cancel{\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}}\sqrt{\frac{\beta}{\alpha}} + \beta\, q_{n+1}\cancel{\left(\frac{\beta}{\alpha}\right)^{-\frac{n}{2}}}\sqrt{\frac{\alpha}{\beta}}$$

- $$\dot{q}_n = -q_n\, 2\sqrt{\alpha\beta} + q_{n-1}\sqrt{\alpha\beta} + q_{n+1}\sqrt{\alpha\beta}$$

$$\sum_n \dot{q}_n z^n = \sum_n -q_n 2\sqrt{\alpha\beta}\cdot z^n + \sum_n q_{n-1}\sqrt{\alpha\beta}\cdot z^n + \sum_n q_{n+1}\sqrt{\alpha\beta}\cdot z^n$$

$$\frac{\partial}{\partial t}\sum_n q_n z^n = -2\sqrt{\alpha\beta}\sum_n q_n z^n + \sum_N q_N\sqrt{\alpha\beta}\, z z^N + \sum_N q_N\sqrt{\alpha\beta}\, z^N\frac{1}{z}$$

$$\frac{\partial}{\partial t}F(z,t) = \left(-2\sqrt{\alpha\beta} + \sqrt{\alpha\beta}\, z + \sqrt{\alpha\beta}\,\frac{1}{z}\right)F(z,t)$$

$$\int\frac{1}{F}\,dF = \int(\qquad)\,dt$$

$$\ln F = (\qquad)\,t \quad\rightarrow\quad F(z,t) = e^{\sqrt{\alpha\beta}\,t\left(-2+z+\frac{1}{z}\right)}\cdot C_0$$

- $F(1,t) = 1 \rightarrow C_0 = 1$

$$\left(\frac{1}{z}+z-2\right) = \frac{1+z^2-2z}{z} = \frac{(z-1)^2}{z}$$

$$F(z,t) = \sum_n P_n(t)\cdot z^n = e^{-\sqrt{\alpha\beta}\,t\,2}\cdot e^{\sqrt{\alpha\beta}\,t\,z}\cdot e^{\sqrt{\alpha\beta}\frac{t}{z}}$$

$$= \sum_n P_n(t)\cdot z^n = e^{-2t\sqrt{\alpha\beta}}\cdot\sum_n\frac{(\sqrt{\alpha\beta}\,t\,z)^n}{n!}\cdot\sum_\ell\frac{(\sqrt{\alpha\beta}\,t)^\ell}{\ell!}\frac{1}{z^\ell}$$

$$= e^{-2t\sqrt{\alpha\beta}}\sum_n\sum_\ell\frac{(\sqrt{\alpha\beta})^{n+\ell}\,t^{n+\ell}}{\ell!\,n!}z^{n-\ell}$$

$n-\ell = k$
$n = k+\ell$

$$= e^{-2t\sqrt{\alpha\beta}}\sum_k\sum_\ell\frac{(\sqrt{\alpha\beta})^{k+2\ell}\,t^{k+2\ell}}{\ell!\,(k+\ell)!}z^k$$

$$F(z,t) = \sum_k\underbrace{\left[\sum_\ell e^{-2t\sqrt{\alpha\beta}}\frac{(\alpha\beta)^{\frac{k}{2}+\ell}\,t^{k+2\ell}}{\ell!\,(k+\ell)!}\right]}_{\equiv\, q_n(t)}z^k$$

$$\left.\frac{\partial F(z,t)}{\partial z}\right|_{z=1} = \sum_k \sum_\ell e^{-zt\sqrt{\alpha\beta}} \frac{(\alpha\beta)^{\frac{k}{2}+\ell}\, t^{k+2\ell}}{\ell!\,(k+\ell)!} k = \langle n\rangle(t)$$

$$\boxed{k\, e^{-2t\sqrt{\alpha\beta}} \sum_k \sum_\ell \frac{(\alpha\beta)^{\frac{k}{2}+\ell}\, t^{k+2\ell}\, k}{\ell!\,(k+\ell)!} = \langle n\rangle}$$

11.

$t=0$ n_0 núcleos activos $\alpha \equiv$ prob. de decaer en la unidad de tiempo de cada núcleo

Estado de la muestra $\equiv$ # n de núcleos activos

Los núcleos decaen $\Rightarrow$ pasamos de $n \to n+1$ en un Δt [ocurre un solo evento en Δt]

a)

$$\dot{P}_n(t) = \sum_m \alpha(m+1)P_{n+1} - \alpha\, m\, P_n$$

$$\dot{P}_n(t) = \underbrace{\alpha(n+1)}_{\omega} P_{n+1} - \underbrace{\alpha\, n}_{\omega} P_n$$

$$\boxed{\frac{\partial P_n(t)}{\partial t} = \alpha(n+1)P_{n+1} - \alpha\, n\, P_n}$$

solo sobreviven dos términos $m=n+1$, $m=n$

Corresponde a que consideramos solo transiciones entre estados contiguos $n+1 \to n \to n-1$

Las transiciones ocurren lo suficientemente lento como para que disminuya de a uno la cantidad de núcleos

b)

$$\sum_n \dot{P}_n(t) s^n = \sum_n \alpha(n+1) P_{n+1} s^n - \alpha\, n\, P_n s^n$$

$$\frac{\partial}{\partial t} \sum_n P_n(t) s^n = \frac{1}{s} \sum_{n'} \alpha\, n' P_{n'} s^{n'} - \sum_n \alpha\, n\, P_n s^n$$

$$\frac{\partial}{\partial t}\left(\sum_n P_n(t) s^n\right) = \left(\frac{1}{s} - 1\right)\alpha \sum_n n\, P_n\, s^n \longrightarrow \frac{\partial}{\partial s} P_n s^n = n\, P_n s^{n-1} = n\, P_n \frac{s^n}{s}$$

$$\frac{\partial}{\partial t} F(s,t) = \left(\frac{1}{s} - 1\right)\alpha\, s \frac{\partial}{\partial s} \sum_n P_n s^n$$

$$\boxed{\frac{\partial}{\partial t} F = \alpha(1-s)\frac{\partial}{\partial s} F}$$

c) En $t=0$ hay n_0 núcleos activos $\to$ $P_n(t=0) = \delta_{n,n_0}$ $\to$

$$F(s,t) = \sum_n P_n(t) s^n \to$$

$$F(s,0) = \sum_n P_n(0) s^n = \sum_n \delta_{n,n_0} \cdot s^n = s^{n_0}$$

$$\langle n(t)\rangle = \sum_n n\, P_n(t) \to \langle n\rangle_{t=0} = \sum_n n\, \delta_{n,n_0} = n_0$$

$$F(s,t) = \phi\left([1-s].e^{-\alpha t}\right) = \phi[\omega(s,t)]$$

$$\cancel{\frac{d\phi}{d\omega}} \cdot \frac{\partial \omega}{\partial t} = \alpha(1-s)\cancel{\frac{d\phi}{d\omega}} \cdot \frac{\partial \omega}{\partial s}$$

$$-(1-s)\alpha . e^{-\alpha t} = -\alpha(1-s).e^{-\alpha t} \Rightarrow$$ cualquier ϕ sirve $\Rightarrow$

tomamos $C.(1-s).e^{-\alpha t} \equiv \phi(s,t) = F(s,t)$

$$F(s,0) = C(1-s) = s^{n_0} \rightarrow C = \frac{s^{n_0}}{1-s}$$

$$\boxed{F(s,t) = s^{n_0}.e^{-\alpha t}}$$

$$\langle n \rangle = \frac{\partial}{\partial s} F(s,t)\Big|_{s=1} = n_0 . s^{n_0-1} . e^{-\alpha t}\Big|_{s=1} = n_0 . e^{-\alpha t} \rightarrow$$

$$\boxed{\langle n \rangle = n_0 . e^{-\alpha t}}$$

$$\sigma = \langle n^2 \rangle - \langle n \rangle^2$$

$\langle n^2 \rangle = \sum_n P_n(t) n^2$; pero no tenemos $P_n(t)$ $\rightarrow$ aunque

$$\langle n^2 \rangle_t = \langle n \rangle_t + \frac{\partial^2}{\partial s^2} F(s,t)\Big|_{s=1}$$

$$\langle n^2 \rangle_t = \langle n \rangle_t + n_0(n_0-1) s^{n_0-2} . e^{-\alpha t}\Big|_{s=1} = \langle n \rangle_t + n_0(n_0-1) e^{-\alpha t}$$

$$\sigma = n_0 e^{-\alpha t} + n_0(n_0-1) e^{-\alpha t} - n_0^2 . e^{-2\alpha t}$$

$$\sigma = \cancel{n_0 e^{-\alpha t}} \; n_0^2 e^{-\alpha t} \cancel{- n_0 e^{-\alpha t}} - n_0^2 e^{-2\alpha t} = \boxed{n_0^2 . e^{-\alpha t}\left(1 - e^{-\alpha t}\right)}$$