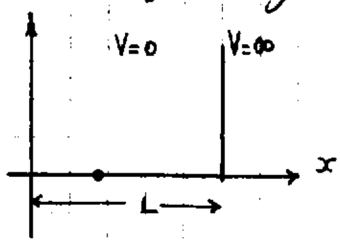
Práctica 5: Simetrías

1.

a) En una caja de longitud L



$$\Psi(x) = \sqrt{\frac{2}{L}} \cdot \text{sen}\left(\frac{n\pi x}{L}\right) \qquad n \in \mathbb{N}$$

$$E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \qquad \text{Una partícula}$$

Proviene de un $H = \frac{p^2}{2m}$

El H total será: $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m}$ $\qquad H\Psi = E_n \Psi$

$$E_1 = \frac{n_1^2 \hbar^2 \pi^2}{2mL^2}, \quad E_2 = \frac{n_2^2 \hbar^2 \pi^2}{2mL^2}, \quad E_3 = \frac{n_3^2 \hbar^2 \pi^2}{2mL^2} \qquad n_1, n_2, n_3 \in \mathbb{N}$$

$$E_n = (n_1^2 + n_2^2 + n_3^2)\frac{\hbar^2 \pi^2}{2mL^2}$$

$$E_1 = \frac{\hbar^2 \pi^2}{2mL^2}(1)$$

$$E_2 = \frac{\hbar^2 \pi^2}{2mL^2}(2)$$

$$E_3 = \frac{\hbar^2 \pi^2}{2mL^2}(3)$$

$$E_4 = \frac{\hbar^2 \pi^2}{2mL^2}(4)$$

	n_1	n_2	n_3	deg · deg spin	deg total
$E_1 = 1$	1 0 0	0 1 0	0 0 1	3 . 2	6
$E_2 = 2$	1 0 1	1 1 0	0 1 1	3 . 4	12
$E_3 = 3$	1	1	1	1 . 8	8
$E_4 = 4$	0 0 2	0 2 0	2 0 0	3 . 2	6
$E_5 = 5$	2 0 0 2 1 1	1 2 1 0 2 0	0 1 2 1 0 2	6 . 4	24
$E_6 = 6$	2 1 1	1 2 1	1 1 2	3 . 8	24
$E_8 = 8$	2 2 0	2 0 2	0 2 2	3 . 4	12
$E_9 = 9$	3 0 0 2 2 1	0 3 0 2 1 2	0 0 3 1 2 2	6 . 2 + 6 . 8	60

3. $|\Phi\rangle$ autoestado de $\hat{A},\hat{B}$ con $\{\hat{A},\hat{B}\}=0$

anticonmutan

[3] $\hat{A}|\Phi\rangle = a|\Phi\rangle$

[2] $\hat{B}|\Phi\rangle = b|\Phi\rangle$

$\hat{A}\hat{B} + \hat{B}\hat{A} = 0$

$\hat{A}\hat{B} = -\hat{B}\hat{A}$ [1]

$AB|\Phi\rangle = b.a\,|\Phi\rangle$

$BA|\Phi\rangle = a\,b\,|\Phi\rangle$

Como $\hat{A},\hat{B}$ son hermíticos $\Rightarrow a,b \in \mathbb{R}$

$$(AB+BA)|\Phi\rangle = (b.a + ab)|\Phi\rangle =$$

Existe una aparente paradoja

$$AB|\Phi\rangle = A\,b|\Phi\rangle = b(A|\Phi\rangle)$$

por [1] $\quad -B(A|\Phi\rangle) = b\,(A|\Phi\rangle)$

$B(A|\Phi\rangle) = -b\,(A|\Phi\rangle) \rightarrow A|\Phi\rangle$ es autoestado de B con autovalor $-b$

$B^2(A|\Phi\rangle) = -b(BA|\Phi\rangle) = -b(-A\,b|\Phi\rangle) = b^2(A|\Phi\rangle)$

$\rightarrow A|\Phi\rangle$ es autoestado de B^2 con autovalor b^2

por [2] $\quad B^2|\Phi\rangle = b^2|\Phi\rangle \rightarrow$

$A|\Phi\rangle = |\Phi\rangle \Rightarrow$

$A^2|\Phi\rangle = A(A|\Phi\rangle) = A(|\Phi\rangle) = |\Phi\rangle$

por [3] $\quad A^2|\Phi\rangle = a^2|\Phi\rangle$

$a^2 = 1 \rightarrow$ $\boxed{a = \pm 1}$

$A(B|\Phi\rangle) = -BA|\Phi\rangle = -a(B|\Phi\rangle)$

$A^2(B|\Phi\rangle) = -a(AB|\Phi\rangle) = a(BA|\Phi\rangle) = a^2(B|\Phi\rangle)$

$A^2|\Phi\rangle = a^2|\Phi\rangle \rightarrow B|\Phi\rangle = |\Phi\rangle$ pero

$B^2|\Phi\rangle = B(B|\Phi\rangle) = B|\Phi\rangle = |\Phi\rangle = b^2|\Phi\rangle$

$\rightarrow b^2 = 1$ $\boxed{b = \pm 1}$

Podemos ilustrar usando π, $\vec{p} = \frac{d\vec{x}}{dt}$

$$\pi^+ \vec{x}\,\pi = -\vec{x}$$

$$\frac{d}{dt}(\pi^+ \vec{x}\,\pi) = -\frac{d\vec{x}}{dt} = -\vec{p}$$

$\pi^+\vec{p}\,\pi = -\vec{p} \rightarrow \pi^+\vec{p}\,\underbrace{\pi\pi^+}_{1} = -\vec{p}\,\pi^+$

$\pi^+\vec{p} + \vec{p}\,\pi^+ = 0 \Rightarrow$ como $\pi^+ = \pi^{-1} = \pi$

$\{\pi,\vec{p}\} = 0$ anticonmutan, luego considerando autoestados de $\vec{p}$:

$\vec{p}|p'\rangle = p'|p'\rangle \rightarrow \pi\,\vec{p}\,|p'\rangle = p'\,\pi|p'\rangle$

$-\vec{p}(\pi|p'\rangle) = p'(\pi|p'\rangle)$

$\vec{p}\,(\pi|p'\rangle) = -p'\,(\pi|p'\rangle)$; pero $-p'$ corresponde a $|-p'\rangle$

$\Rightarrow$ $\boxed{\pi|p'\rangle = |-p'\rangle}$

$\pi(\pi|p'\rangle) = \pi(|-p'\rangle) = |p'\rangle \rightarrow \pi^2|p'\rangle = \mathbb{1}|p'\rangle = |p'\rangle$

4.

a) $\mathcal{Y}_{\ell=0}^{j=1/2, m=1/2} = +\sqrt{\dfrac{0+1/2+1/2}{2.0+1}}\, Y_0^0(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\dfrac{0-1/2+1/2}{2.0+1}}\, Y_0^1(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix}$

$\mathcal{Y}_{\ell=0}^{j=\frac{1}{2}, m=\frac{1}{2}} = Y_0^0(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} = \dfrac{1}{\sqrt{4\pi}}\begin{pmatrix}1\\0\end{pmatrix}$

b) $\overleftrightarrow{\sigma}\cdot\vec{x}$ es la nomenclatura para las matrices de Pauli y se sobreentiende que esto significa

$\overleftrightarrow{\sigma}_x x + \overleftrightarrow{\sigma}_y y + \overleftrightarrow{\sigma}_z z$

$(\overleftrightarrow{\sigma}\cdot\vec{x})\mathcal{Y}_{\ell=0}^{j=\frac{1}{2}, m=\frac{1}{2}} = \left[\begin{pmatrix}0&1\\1&0\end{pmatrix}x + \begin{pmatrix}0&-i\\i&0\end{pmatrix}y + \begin{pmatrix}1&0\\0&-1\end{pmatrix}z\right]\dfrac{1}{\sqrt{4\pi}}\begin{pmatrix}1\\0\end{pmatrix}$

$= \left[x\begin{pmatrix}0\\1\end{pmatrix} + y\begin{pmatrix}0\\i\end{pmatrix} + z\begin{pmatrix}1\\0\end{pmatrix}\right]\dfrac{1}{\sqrt{4\pi}}$

$= \left[r.\cos\phi.\mathrm{sen}\theta\begin{pmatrix}0\\1\end{pmatrix} + r.\mathrm{sen}\phi.\mathrm{sen}\theta\begin{pmatrix}0\\i\end{pmatrix} + r.\cos\theta\begin{pmatrix}1\\0\end{pmatrix}\right]Y_0^0$

$= r.Y_0^0\begin{pmatrix}\cos\theta\\ \cos\phi.\mathrm{sen}\theta + i.\mathrm{sen}\phi.\mathrm{sen}\theta\end{pmatrix} = r.Y_0^0\begin{pmatrix}\cos\theta\\ \mathrm{sen}\theta.e^{i\phi}\end{pmatrix}$

$\boxed{(\sigma\cdot\vec{x})\mathcal{Y}_{\ell=0}^{j=\frac{1}{2}, m=\frac{1}{2}} = r.Y_0^0\begin{pmatrix}Y_1^0.\sqrt{4\pi/3}\\ -Y_1^1.\sqrt{8\pi/3}\end{pmatrix}}$

c) $\vec{S}\cdot\vec{x}$ es un escalar $\Rightarrow$ no cambia ante rotaciones $\quad(\hat{\vec{S}} = \hbar\overleftrightarrow{\sigma})$

$\pi^+\vec{S}\cdot\vec{x}\,\pi = \pi^+\vec{S}.\pi(\pi^+\vec{x}\,\pi) = \pi^+\vec{S}\pi\cdot(-\vec{x}) = -\vec{S}\cdot\vec{x}$

$\vec{S}\cdot\vec{x}$ transforma como pseudoescalar ante paridad

6.

a) $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ onda plana en 3D (partícula libre)

$$\phi(x,t) = e^{i\left(\frac{\vec{p}\cdot\vec{x}}{\hbar} - \frac{E}{\hbar}t\right)} \qquad \hbar\vec{k} = \vec{p} \qquad \hbar\omega = E$$

$$\phi^*(x,-t) = e^{-i\left(\frac{\vec{p}\cdot\vec{x}}{\hbar} - \frac{E}{\hbar}(-t)\right)} = e^{-i\left(\frac{\vec{p}\cdot\vec{x}}{\hbar} + \frac{E}{\hbar}t\right)} = e^{i\left(\frac{(-\vec{p})\cdot\vec{x}}{\hbar} - \frac{E}{\hbar}t\right)}$$

Comparando ϕ^* con ϕ se ve que tienen la dirección de $\vec{p}$ invertida.

b)

$$(\vec{\sigma}\cdot\hat{n})\,\chi(\hat{n}) = +1\,\chi(\hat{n})$$

$$\sigma_x . n_x + \sigma_y . n_y + \sigma_z\, n_z$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}n_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}n_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}n_z = \begin{pmatrix}n_z & n_x - i n_y\\ n_x + i n_y & -n_z\end{pmatrix} = \begin{pmatrix}\cos\theta & \cos\phi . \operatorname{sen}\theta - i\,\operatorname{sen}\phi . \operatorname{sen}\theta\\ \cos\phi . \operatorname{sen}\theta + i . \operatorname{sen}\phi . \operatorname{sen}\theta & -\cos\theta\end{pmatrix}$$

$$(\vec{\sigma}\cdot\hat{n}) = \begin{pmatrix}\cos\beta & e^{-i\alpha}\operatorname{sen}\beta\\ e^{i\alpha} . \operatorname{sen}\beta & -\cos\beta\end{pmatrix}$$

$$\begin{pmatrix}\cos\beta & \operatorname{sen}\beta\, e^{-i\alpha}\\ \operatorname{sen}\beta\, e^{i\alpha} & -\cos\beta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} - \mathbb{1}\begin{pmatrix}a\\ b\end{pmatrix} = 0 = (\vec{\sigma}\cdot\hat{n} - \mathbb{1})\begin{pmatrix}a\\ b\end{pmatrix} = 0$$

$$\begin{pmatrix}\cos\beta - 1 & \operatorname{sen}\beta . e^{-i\alpha}\\ \operatorname{sen}\beta . e^{i\alpha} & -\cos\beta - 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0$$

$$-(\cos\beta - 1)(\cos\beta + 1) - \operatorname{sen}^2\beta = -\cos^2\beta + 1 - \operatorname{sen}^2\beta = 0$$

$$\begin{matrix}e^{-i\frac{\alpha}{2}}\\ e^{i\frac{\alpha}{2}}\end{matrix}\begin{pmatrix}e^{i\alpha/2}(\cos\beta - 1) & \operatorname{sen}\beta . e^{-i\alpha/2}\\ e^{i\alpha/2} . \operatorname{sen}\beta & -(\cos\beta + 1)e^{-i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0$$

$$a = \frac{-(1+\cos\beta)\, e^{-i\alpha/2}}{e^{+i\frac{\alpha}{2}} . \operatorname{sen}\beta}\, b \qquad a = \frac{-\operatorname{sen}\beta . e^{-i\alpha/2}}{e^{i\alpha/2} . (\cos\beta - 1)} . b = \frac{\cancel{2}\operatorname{sen}(\beta/2)\cos(\beta/2)\, e^{-i\alpha/2}}{e^{i\alpha/2} . \cancel{2}\operatorname{sen}^2\left(\frac{\beta}{2}\right)} . b$$

$$a = \frac{-\cancel{2}\cos^{\cancel{2}}(\beta/2)\, e^{-i\alpha}}{\cancel{2} . \operatorname{sen}\beta/2 \cdot \cancel{\cos\beta/2}}\, b \qquad a = \frac{b}{e^{i\alpha} . \tan(\beta/2)} \quad \rightarrow \quad \chi = \begin{pmatrix}\frac{e^{-i\alpha} . \cos(\beta/2)}{\operatorname{sen}(\beta/2)}\, b\\ b\end{pmatrix}$$

$$a^2 + b^2 = 1$$

$$b^2\left(\left(\frac{e^{-i2\alpha} . \cos^2(\beta/2)}{\operatorname{sen}^2(\beta/2)}\right) + 1\right) = 1$$

$$b^2\left(e^{-i\alpha 2} . \cos^2(\beta/2) + \operatorname{sen}^2(\beta/2)\right) = \operatorname{sen}^2\left(\frac{\beta}{2}\right)$$

$$\chi = \begin{pmatrix}\dfrac{e^{-i\alpha}\cos(\beta/2)}{\left[e^{-i\alpha 2} . \cos^2(\beta/2) + \operatorname{sen}^2(\beta/2)\right]^{1/2}}\\[2ex] \dfrac{\operatorname{sen}(\beta/2)}{\left[e^{-i\alpha 2} . \cos^2(\beta/2) + \operatorname{sen}^2(\beta/2)\right]^{1/2}}\end{pmatrix}$$

$$\left(e^{-i\alpha}\right)^{1/2}\left(\cos^2 e^{-i\alpha} + \operatorname{sen}^2 e^{+i\alpha}\right)^{1/2}$$

$$\chi(\hat{n}) = \frac{1}{\left(\cos^2\left(\frac{\beta}{2}\right)e^{-i\alpha} + \operatorname{sen}^2\left(\frac{\beta}{2}\right)e^{i\alpha}\right)^{1/2}}\begin{pmatrix}\cos(\beta/2) . e^{-i\frac{\alpha}{2}}\\ \operatorname{sen}(\beta/2) . e^{i\frac{\alpha}{2}}\end{pmatrix}$$

$$-i\,\sigma_y\,\chi^*(\hat{n}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{1}{\left[\cos^2\left(\frac{\beta}{2}\right)e^{i\alpha} + \operatorname{sen}^2\left(\frac{\beta}{2}\right)e^{-i\alpha}\right]^{1/2}} \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) e^{\frac{i\alpha}{2}} \\ \operatorname{sen}\left(\frac{\beta}{2}\right) e^{-\frac{i\alpha}{2}} \end{pmatrix}$$

$$-i\,\sigma_y\,\chi^*(\hat{n}) = \frac{1}{\left[\cos^2\left(\frac{\beta}{2}\right)e^{i\alpha} + \operatorname{sen}^2\left(\frac{\beta}{2}\right)e^{-i\alpha}\right]^{1/2}} \begin{pmatrix} -\operatorname{sen}\left(\frac{\beta}{2}\right) e^{-\frac{i\alpha}{2}} \\ \cos\left(\frac{\beta}{2}\right) e^{\frac{i\alpha}{2}} \end{pmatrix}$$

Pero sea $\hat{n} \to -\hat{n} \Rightarrow \left.\begin{array}{l} \beta \to \pi - \beta \\ \alpha \to \alpha + \pi \end{array}\right\} \Rightarrow$

$$\chi(-\hat{n}) = \frac{1}{\left[\cos^2\left(\frac{\pi-\beta}{2}\right) e^{-i\alpha} e^{-i\pi} + \operatorname{sen}^2\left(\frac{\pi-\beta}{2}\right) e^{i\alpha} e^{i\pi}\right]^{1/2}} \begin{pmatrix} \cos\left(\frac{\pi-\beta}{2}\right) e^{-\frac{i\alpha}{2}} e^{\frac{i\pi}{2}} \\ \operatorname{sen}\left(\frac{\pi-\beta}{2}\right) e^{\frac{i\alpha}{2}} e^{\frac{i\pi}{2}} \end{pmatrix}$$

$$\chi(-\hat{n}) = \frac{1}{\left[+\operatorname{sen}^2\left(\frac{\beta}{2}\right) e^{-i\alpha}(-1) + \cos^2\left(\frac{\beta}{2}\right) e^{i\alpha}(-1)\right]^{1/2}} \begin{pmatrix} \operatorname{sen}\left(\frac{\beta}{2}\right) \cdot e^{-\frac{i\alpha}{2}} \cdot i \\ \cos\left(\frac{\beta}{2}\right) \cdot e^{\frac{i\alpha}{2}} \cdot i \end{pmatrix}$$

$$\chi(-\hat{n}) = \frac{1}{\not{i}\left(\cos^2\left(\frac{\beta}{2}\right)e^{i\alpha} + \operatorname{sen}^2\left(\frac{\beta}{2}\right)e^{-i\alpha}\right)^{1/2}}\, \not{i} \begin{pmatrix} \operatorname{sen}(\beta/2)\, e^{-i\alpha/2} \\ \cos(\beta/2)\, e^{i\alpha/2} \end{pmatrix}$$

$$\Rightarrow \boxed{\chi(-\hat{n}) = -i\,\sigma_y\,\chi^*(\hat{n})}$$

7.

a) H invariante ante inversión temporal $\Rightarrow$

$$\Theta^{+} H \Theta = H \to \Theta\Theta^{+} H \Theta = H\Theta = \Theta H \to [H, \Theta] = 0$$

$$H(\Theta|n\rangle) = \Theta H|n\rangle = \Theta E_n|n\rangle = E_n^* \Theta|n\rangle = E_n(\Theta|n\rangle)$$, con $|n\rangle$ autoestado de H

$$\to \Theta|n\rangle = e^{i\delta}|n\rangle \quad (1) \quad (\delta \text{ fase inicial libre})$$

$\Theta|n\rangle$, $|n\rangle$ son autoestados de H con igual autovalor $E_n \Rightarrow$ vale (1)
Luego la función de onda será:

$$\langle\vec{x}|\Theta|n\rangle = \langle\vec{x}|e^{i\delta}|n\rangle = e^{i\delta}\langle\vec{x}|n\rangle$$

pero sabemos que $\langle\vec{x}|\tilde{\alpha}\rangle = \langle\vec{x}|\alpha\rangle^* \Rightarrow$

$$\langle\vec{x}|\Theta|n\rangle = e^{i\delta}\langle\vec{x}|n\rangle = \langle\vec{x}|n\rangle^*$$

Si tomo $\delta = 0 \Rightarrow \langle\vec{x}|n\rangle = \langle\vec{x}|n\rangle^*$

$$\psi(\vec{x}) = \psi^*(\vec{x})$$

Resulta función de onda elegida real.

b) $$\Psi(\vec{x},0) = e^{i\vec{p}\cdot\vec{x}/\hbar} = \langle \vec{x} \mid n, t=0\rangle$$

Un estado de onda plana refiere al hamiltoniano $H = \frac{P^2}{2m}$ del cual es autoestado. Asimismo $[H, \Theta] = 0$ y entonces $|p\rangle$ son autoestados de H, pero son degenerados porque la autoenergía $\frac{p^2}{2m}$ le corresponde al autoestado $|p\rangle$ y también al $|-p\rangle$

$\Rightarrow$ no estamos cumpliendo todas las hipótesis del teorema para asegurar que Ψ puede elegirse $\mathbb{R}$ con la fase apropiada.

Asimismo $\left.\begin{array}{l}\Theta|p\rangle = |-p\rangle \\ \Theta|-p\rangle = |p\rangle\end{array}\right\}$ con lo cual no son autoestados de Θ los autoestados de H.

8.

$$\phi(\vec{p}') = \langle \vec{p}' | \alpha\rangle$$

(al hacer actuar Θ opero sobre todo lo ubicado a su derecha)

$$\langle \vec{p}'|(\Theta|\alpha\rangle) = \int dp'' \langle \vec{p}' | \Theta | \vec{p}''\rangle\langle \vec{p}''|\alpha\rangle = \int dp'' \langle \vec{p}'|-\vec{p}''\rangle\langle \vec{p}''|\alpha\rangle^*$$

$$= \int dp'' \,\delta(\underbrace{\vec{p}'+\vec{p}''}_{=[\vec{p}''-(-\vec{p}')]})\langle \vec{p}''|\alpha\rangle^* = \langle -\vec{p}'|\alpha\rangle^* = \phi^*(-\vec{p}')$$

$\Rightarrow$ La función de onda del estado inverso temporal es $\phi^*(-\vec{p}')$

9.

a)

$$\Theta\,\mathcal{D}(R)\,|j,m\rangle$$

$$\Theta\, e^{i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi}\,|j,m\rangle$$

$$\Theta \sum_{l=0}^{\infty} (-1)^l \left(i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi\right)^l \frac{1}{l!}\,|j,m\rangle$$

Queremos ver si conmuta o anticonmuta con reversión de movimiento

$i^3 = -i$

$l=0$ $\quad 1\;\Theta|j,m\rangle$

$l=1$ $\quad -\frac{i}{\hbar}(-\vec{J}\cdot\hat{n})\phi\;\Theta|j,m\rangle = \frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\phi\;\Theta|j,m\rangle$

$l=2$ $\quad \left(\frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\phi\right)\left(\frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\phi\right)\frac{1}{2!}\;\Theta|j,m\rangle =$

$l=3$ $\quad (\quad)(\quad)(\quad)\frac{1}{3!}\;\Theta|j,m\rangle = \left(i\frac{\vec{J}\cdot\hat{n}\phi}{\hbar}\right)^3\frac{1}{3!}$

$\therefore$ Θ no cambia el operador al traspasarlo $\Rightarrow$ en realidad es más prolijo aplicarlo así:

$l=1$ $\quad \Theta\left(i\frac{\vec{J}\cdot\hat{n}\,\phi}{\hbar}\right)\Theta^{-1} = -i\frac{(-\vec{J}\cdot\hat{n})}{\hbar}\phi\;\Theta\Theta^{-1} = \left(i\frac{\vec{J}\cdot\hat{n}\,\phi}{\hbar}\right)$

$l=2$ $\quad \Theta\; i^2\frac{(\vec{J}\cdot\hat{n})^2\phi^2}{\hbar^2}\;\Theta^{-1} = i^2\frac{(-\vec{J}\cdot\hat{n})(-\vec{J}\cdot\hat{n})\phi^2}{\hbar^2}\;\Theta\Theta^{-1} = \left(\frac{i^2(\vec{J}\cdot\hat{n})^2\phi^2}{\hbar^2}\right)$

$l=3$ $\quad \Theta\; i^3\frac{(\vec{J}\cdot\hat{n})^3\phi^3}{\hbar^3}\;\Theta^{-1} = i\frac{(-\vec{J}\cdot\hat{n})(\vec{J}\cdot\hat{n})^2\phi^3}{\hbar^3}\;\Theta\Theta^{-1} = \left(\frac{i^3(\vec{J}\cdot\hat{n})^3\phi^3}{\hbar^3}\right)$

Puede verse, entonces, que Θ no produce cambios en el operador $\mathcal{D}(R)$ $\Rightarrow$

$$\Theta\mathcal{D}(R) = \mathcal{D}(R)\Theta \rightarrow [\Theta, \mathcal{D}(R)] = 0$$

$$\Theta\mathcal{D}(R)|j,m\rangle = \mathcal{D}(R)\Theta|j,m\rangle = \mathcal{D}(R)\, i^{2m}|j,-m\rangle$$

$$\boxed{\Theta\mathcal{D}(R)|j,m\rangle = e^{-i\frac{\vec{J}\cdot\hat{n}}{\hbar}*} . i^{2m} |j,-m\rangle}$$

b)

$$\mathcal{D}(R)|j,m\rangle = \sum_{m'} |j,m'\rangle\langle j,m'|\mathcal{D}(R)|j,m\rangle$$

$$\langle j,m'|\mathcal{D}(R)|j,m\rangle = d^{j}_{m'm} \rightarrow$$

$$d^{j*}_{m'm} = \left(\langle j,m'|\mathcal{D}(R)|j,m\rangle\right)^*$$

$$d^{j*}_{m'm} = \left(\langle j,m'|\mathcal{D}(R)\Theta\Theta^{-1}|j,m\rangle\right)^*$$

$$= \left(\langle j,m'|\mathcal{D}(R)\Theta\, i^{2m}|j,-m\rangle\right)^*$$

$$\text{DC}\downarrow \Theta|j,m\rangle = i^{2m}|j,-m\rangle \underset{\text{DC}}{\searrow}\longrightarrow \Theta|j,-m\rangle = i^{-2m}|j,m\rangle$$

$$\langle j,m|\Theta^{-1} = \langle j,-m|-i^{2m}$$

$$\left(\langle j,m'|\Theta\mathcal{D}(R)(-1)^m|j,-m\rangle\right)^*$$

$$\sum_{m''}\left(\langle j,m'|\Theta|j,-m''\rangle\langle j,-m''|\mathcal{D}(R)(-1)^m|j,-m\rangle\right)^*$$

$$\left(\langle j,m'|i^{-2m''}|j,m''\rangle\right)^*\left(\langle j,-m|\mathcal{D}^*_{(R)}(-1)^m|j,-m''\rangle\right)^*$$

$$\left(\sum_{m''}(-1)^{-m''}\langle j,m'|j,m''\rangle(-1)^m\langle j,-m''|\mathcal{D}(R)|j,-m\rangle\right.$$

$$\sum_{m''}(-1)^{m-m''}\,\delta_{m'm''}\left(\langle j,-m'|\mathcal{D}(R)|j,-m\rangle\right)$$

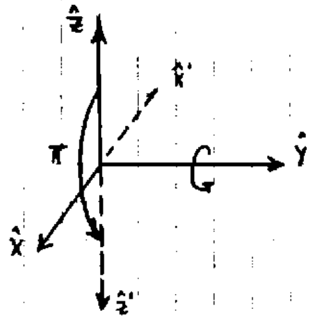
$$(-1)^{m-m'}\langle j,-m'|\mathcal{D}(R)|j,-m\rangle$$

$$\Rightarrow \boxed{D^{j*}_{m'm} = (-1)^{m-m'} D^{j}_{-m'-m}}$$

c) Podemos usar $\Theta = e^{i\delta}e^{-i\pi J_y/\hbar}K \rightarrow$

$$\Theta|j,m\rangle = e^{i\delta}e^{-i\pi J_y/\hbar}K|j,m\rangle = e^{i\delta}K.e^{\frac{i\pi J_y}{\hbar}}|j,m\rangle$$

físicamente veo que $e^{i J_y \pi/\hbar}$ rotará al ket $\Rightarrow$



$$e^{i J_y \pi/\hbar} |j,m\rangle = |j,-m\rangle \Rightarrow$$

$$\Theta |j,m\rangle = e^{i\delta} K |j,-m\rangle = e^{i\delta} |j,-m\rangle$$

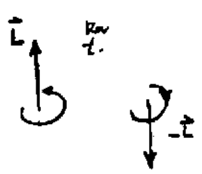
10.

Partícula sin spin $\rightarrow$ $H|n\rangle = E_n |n\rangle$, no degenerado

$$\langle n | \vec{L} | n\rangle = \langle n | \Theta \vec{L} \Theta^{-1} | n\rangle = -\langle n | \Theta \vec{L} \Theta | n\rangle$$

, como hay invarianza ante inversión temporal $\Rightarrow$

$$\Theta H \Theta^{-1} = H \rightarrow [\Theta, H] = 0 \Rightarrow$$



$$\Theta H |n\rangle = \Theta E_n |n\rangle = E_n \Theta |n\rangle \Rightarrow \Theta|n\rangle = e^{i\delta}|n\rangle \Rightarrow$$

$\searrow$ DC $\langle n | \Theta^{+} = \langle n | e^{-i\delta}$

$$\langle n | \vec{L} | n\rangle = -\langle n | e^{-i\delta} L \, e^{i\delta} | n\rangle$$

$$\langle n | \vec{L} | n\rangle = -\langle n | \vec{L} | n\rangle \Rightarrow \boxed{\langle \vec{L} \rangle = 0}$$

$$e^{i\delta}\langle \vec{x} | n\rangle = \sum_{l}\sum_{m} F_{lm}(r)\, Y_l^m(\theta,\phi)\, e^{i\delta}$$

$\parallel$

$$\langle \vec{x} | n\rangle^{*} = \sum_{l}\sum_{m} F^{*}_{lm}(r)\, Y_l^{m*}(\theta,\phi)$$

$$\sum_{l}\sum_{m} F^{*}_{lm}(r)(-1)^m\, Y_l^{-m}(\theta,\phi) = e^{i\delta}\sum_{l}\sum_{m} F_{lm}(r)\, Y_l^m(\theta,\phi)$$

$$\sum_{l,m} e^{i\alpha} F_{lm}(r)\, Y_l^m(\theta,\phi) = \sum_{l,m} F^{*}_{lm}(r)\,(-1)^m\, Y_l^{-m}(\theta,\phi)$$

$$\int Y_{l'}^{m'*} \sum_{l,m} e^{i\alpha} F_{lm}(r)\, Y_l^m(\theta,\phi)\, d\Omega = \int Y_{l'}^{m'*} \sum_{l,m} F^{*}_{lm}(r)(-1)^m\, Y_l^{-m}(\theta,\phi)\, d\Omega$$

$$\sum_{l,m} e^{i\alpha} F_{lm}(r)\, \delta_{mm'}\, \delta_{ll'} = \sum_{l,m} F^{*}_{lm}(r)\,(-1)^m\, \delta_{m',-m}\, \delta_{ll'}$$

$$e^{i\alpha} F_{l'm'}(r) = F^{*}_{lm'}(r)\,(-1)^{-m'}$$

$$\boxed{e^{i\alpha} F_{l'm'}(r)\,(-1)^{m'} = F^{*}_{l',-m'}(r)}$$

11. hamiltoniano de un sistema de spin 1

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

Sistema de spin = 1 $\Rightarrow$ $j = 1 \rightarrow -j \leq m \leq j \rightarrow m = -1, 0, +1$

← Es un sistema de algún momento angular fijo y punto.

$$S_z |j,m\rangle = \hbar m |j,m\rangle$$

$$S^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$\langle j,m'| S_z |j,m\rangle = \hbar m\, \delta_{m'm} = \hbar m \mathbb{1}$$

$$S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2} \rightarrow \langle j,m'| S_x |j,m\rangle = \frac{1}{2}\left(\langle j,m'| S_+ |j,m\rangle + \langle j,m'| S_- |j,m\rangle\right)$$

$$\langle S_x\rangle = \frac{1}{2}\left(\langle m'| \hbar\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle + \langle m'| \hbar\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle\right)$$

$$\frac{\hbar}{2}$$

$$S_x \doteq \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right]$$

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

, $\Rightarrow$ en forma ídem será:

$$\langle S_y\rangle \doteq \frac{\hbar}{2i}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right]$$

$$S_y \doteq \frac{-\hbar}{\sqrt{2}} i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$S_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$S_y^2 = -\frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -\frac{\hbar^2}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H = \hbar^2 \left[\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix} + B \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

$$(H - \lambda \mathbb{1}) |\lambda\rangle = 0$$

$$\left| \begin{pmatrix} \hbar^2 A - 1\lambda & 0 & \hbar^2 B \\ 0 & -\lambda & 0 \\ \hbar^2 B & 0 & \hbar^2 A - 1\lambda \end{pmatrix} \right| = (\hbar^2 A - \lambda)^2(-\lambda) - (\hbar^2 B)^2(-\lambda) = 0$$

$\lambda_1 = 0$ ←

$$(\hbar^2 A)^2 - 2\hbar^2 A \lambda + \lambda^2 - (\hbar^2 B)^2 = 0$$

$$\frac{2\hbar^2 A \pm \sqrt{4(\hbar^2 A)^2 - 4\left[(\hbar^2 A)^2 - (\hbar^2 B)^2\right]}}{2}$$

$$\lambda_{2,3} = \frac{2\hbar^2 A \pm 2\hbar^2 B}{2} = \hbar^2 (A \pm B)$$

* $\lambda_1 = 0$ →

$$\begin{pmatrix} \hbar^2 A & 0 & \hbar^2 B \\ 0 & 0 & 0 \\ \hbar^2 B & 0 & \hbar^2 A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A.x + Bz = 0$

$-\frac{A}{B} x = z$

$Bx + A.z = 0$

$z = -\frac{B x}{A}$

$\rightarrow \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$

* $\lambda_2 = \hbar^2 (A+B)$

$-\hbar^2 B x + \hbar^2 B z = 0$

$x = z$

$-\hbar^2 (A+B) y = 0$

$\begin{pmatrix} x \\ 0 \\ x \end{pmatrix}$

* $\lambda_3 = \hbar^2 (A-B)$

$\hbar^2 B x + \hbar^2 B z = 0$ → $\begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}$

$-x = z$

$\lambda = 0 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv |1,0\rangle$ (J, m)

$\lambda = \hbar^2 (A+B) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = \hbar^2 (A-B) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$|n; \lambda = 0\rangle = |1,0\rangle$

$|n; \lambda = \hbar^2(A+B)\rangle = \frac{1}{\sqrt{2}} (|1,+\rangle + |1,-\rangle)$

$|n; \lambda = \hbar^2(A-B)\rangle = \frac{1}{\sqrt{2}} (|1,+\rangle - |1,-\rangle)$

← autoestados normalizados del hamiltoniano

$\frac{1}{i^2} = \frac{1}{-1}$

$$\Theta |n, \lambda_2\rangle = \frac{1}{\sqrt{2}} \left(i^2 |1,-\rangle + i^{-2} |1,+\rangle \right) = \frac{1}{\sqrt{2}} \left(-|1,-\rangle + - |1,+\rangle \right) = - |n; \lambda_2\rangle$$

$$\Theta |n, \lambda_3\rangle = \frac{1}{\sqrt{2}} \left(i^2 |1,-\rangle - i^{-2} |1,+\rangle \right) = \frac{1}{\sqrt{2}} \left(-|1,-\rangle + |1,+\rangle \right) = + |n; \lambda_3\rangle$$

$$\Theta |n, \lambda_1\rangle = |1,0\rangle\, i^0 = |1,0\rangle$$

Para ver si un hamiltoniano es invariante ante inversión temporal hay que evaluar

$$\Theta H \Theta^{-1} = \Theta\left(AS_z^2 + B(S_x^2 - S_y^2)\right)\Theta^{-1}$$

$$= \Theta AS_z^2 \Theta^{-1} + \Theta B S_x^2 \Theta^{-1} - \Theta B S_y^2 \Theta^{-1}$$

$$= A\, \Theta S_z \Theta^{-1}\Theta S_z \Theta^{-1} + B\, \Theta S_x \Theta^{-1}\Theta S_x \Theta^{-1} - B\, \Theta S_y \Theta^{-1}\Theta S_y \Theta^{-1}$$

, donde A, B han salido afuera porque son reales porque son números en los autovalores del hamiltoniano y estos últimos son reales

Ahora $\Theta \vec{J} \Theta^{-1} = -\vec{J} \Rightarrow$

$\Theta J_{x,y,z} \Theta^{-1} = -J_{x,y,z} \Rightarrow$

$$= A(-S_z)(-S_z) + B(S_x)(-S_x) - B(-S_y)(S_y)$$

$$\Theta H \Theta^{-1} = AS_z^2 + B(S_x^2 - S_y^2) = H$$

∴ el hamiltoniano es invariante ante inversión temporal