Práctica 4: Suma de Momentos Angulares y Teorema de Wigner-Eckart

1. a) Partícula con spin $1/2$ $\quad l=1$

estados $\{|l,s,m_l,m_s>\}$ $\qquad -l \le m_l \le +l$

$-s \le m_s \le +s$

$l=1 \rightarrow \quad m_l=-1,0,1$

$s=1/2 \rightarrow \quad m_s=-1/2,+1/2$

$\{|-1,-1/2>, |0,-1/2>, |1,-1/2>, |-1,1/2>, |0,1/2>, |1,1/2>\}$

En términos de estos $\{|\overset{\uparrow}{\underset{L^2}{l}},\overset{\uparrow}{\underset{S^2}{s}},\overset{\uparrow}{\underset{L_z}{m_l}},\overset{\uparrow}{\underset{S_z}{m_s}}>\}$ queremos ver la expresión en la base $\{|\underset{L^2}{l},\underset{S^2}{s},\underset{J^2}{j},\underset{J_z}{m_j}>\}$

J_{max} corresponde a $l_{max}+s_{max} \rightarrow \quad J_{max}=1+1/2=3/2 \Rightarrow |J_{max},m_{max}>=|3/2,3/2>$

$m_{J_{max}}$ " " $m_{l_{max}}+m_{s_{max}} \rightarrow \quad m_{J_{max}}=1+1/2=3/2$

L_z, S_z en la misma dirección

$|l,s,j_{max},m_{j_{max}}> = \boxed{|1,1/2,1,1/2> = |3/2,3/2>}$

b) $J_- = L_- + S_- = L_x - iL_y + S_x - iS_y$

$$J_-|j,m> = \sqrt{(J+m)(J-m+1)}\,\hbar\,|j,m-1>$$

$J_{max}=3/2$

$$J_-|j=3/2,m=3/2> = \sqrt{3}\,\hbar\,|3/2,1/2> =$$

$$J_-|3/2,1/2> = \underbrace{\sqrt{2.2}}_{=2}\,\hbar\,|3/2,-1/2> = 2\hbar\,|3/2,-1/2>$$

$$J_-|3/2,-1/2> = \underbrace{\sqrt{(1)3}}_{=\sqrt{3}}\,\hbar\,|3/2,-3/2> = \sqrt{3}\hbar\,|3/2,-3/2>$$

$\therefore \quad J_-|m_l,m_s> = (L_- + S_-)|m_l,m_s> \Rightarrow$

$$\underset{\{|l,s,j,m>\}}{J_-|3/2,3/2>} = \underset{\{|l,s,m_l,m_s>\}}{L_- + S_-|1,1/2>} = (L_- + S_-)|\overset{l}{1},\overset{s}{1/2},\overset{ml}{1},\overset{ms}{1/2}>$$

$$\sqrt{3}\hbar\,|3/2,1/2> = \hbar\sqrt{2}\,|1,1/2,0,1/2> + \hbar\,|1,1/2,1,-1/2>$$

$$|3/2,1/2> = \sqrt{2/3}\,|1,1/2,0,1/2> + \sqrt{1/3}\,|1,1/2,1,-1/2>$$

$$2\hbar\,|3/2,-1/2> = \sqrt{2/3}\,\hbar\sqrt{2}\,|1,1/2,-1,1/2> + \sqrt{1/3}\,\hbar\sqrt{2}\,|1,1/2,0,-1/2>$$
$$+\sqrt{2/3}\,\hbar\,|1,1/2,0,-1/2> =$$

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$$|3/2,-1/2> = \frac{\cancel{2}}{\sqrt{3}}\cdot\frac{1}{\cancel{2}}\,|1,1/2,-1,1/2> + \sqrt{\frac{2}{3}}\cdot\frac{1}{\cancel{2}}\,|1,1/2,0,-1/2>\cancel{2}$$

$$\sqrt{3}\hbar\,|3/2,-3/2> = \sqrt{2/3}\,\hbar\,\sqrt{2}\,|1,1/2,-1,-1/2> + \sqrt{1/3}\,\hbar\,|1,1/2,-1,-1/2>$$

$$|3/2,-3/2> = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{2}}{\sqrt{3}}\,|1,1/2,-1,-1/2> + \frac{1}{\sqrt{3}\sqrt{3}}\,|1,1/2,-1,-1/2>$$

$$|3/2,-3/2> = |1,1/2,-1,-1/2>$$

$$\boxed{\begin{aligned}
|3/2,3/2\rangle &= |1,1/2,1,1/2\rangle \\
|3/2,1/2\rangle &= \sqrt{2/3}\,|1,1/2,0,1/2\rangle + \sqrt{1/3}\,|1,1/2,1,-1/2\rangle \\
|3/2,-1/2\rangle &= \sqrt{1/3}\,|1,1/2,-1,1/2\rangle + \sqrt{2/3}\,|1,1/2,0,-1/2\rangle \\
|3/2,-3/2\rangle &= |1,1/2,-1,-1/2\rangle
\end{aligned}}$$

c)

$$|j_{max}-1, j_{max}-1\rangle = |3/2-1, 3/2-1\rangle = |\underset{j}{1/2}, \underset{m}{1/2}\rangle$$

será combinación lineal de los estados que tengan $m = m_\ell + m_s = 1/2$

$$|1/2,1/2\rangle = A\,|1,1/2,0,1/2\rangle + B\,|1,1/2,1,-1/2\rangle$$

$$\langle 3/2,1/2|1/2,1/2\rangle = 0 = \left(\sqrt{2/3}\,\langle 1,1/2,0,1/2| + \sqrt{1/3}\,\langle 1,1/2,1,-1/2|\right)\left(A\,|1,1/2,0,1/2\rangle + B\,|1,1/2,1,-1/2\rangle\right)$$

$$A\frac{\sqrt{2}}{\sqrt{3}} + B\frac{1}{\sqrt{3}} = 0 \rightarrow B = -\sqrt{2}A \rightarrow$$

$$A^2+B^2 = A^2+2A^2 = 3A^2 = 1$$

$$A = \frac{1}{\sqrt{3}}$$

$$B = -\sqrt{\frac{2}{3}}$$

$$\boxed{|1/2,1/2\rangle = 1/\sqrt{3}\,|1,1/2,0,1/2\rangle - \sqrt{2/3}\,|1,1/2,1,-1/2\rangle}$$

d)

$$|1/2, m\rangle \quad \xrightarrow[\text{serán estos}]{-1/2 \le m \le 1/2} \quad |1/2,-1/2\rangle \;;\; |1/2,+1/2\rangle$$

→ ya está del paso anterior

$$J_-|1/2,1/2\rangle = \hbar\,|1/2,-1/2\rangle$$

$$J_-|1/2,-1/2\rangle = 0$$

$$(L_- + S_-)\left(1/\sqrt{3}\,|0,1/2\rangle - \sqrt{2/3}\,|1,-1/2\rangle\right) = \hbar\,|1/2,-1/2\rangle$$

$$\sqrt{1/3}\,\hbar\sqrt{2}\,|-1,1/2\rangle - \sqrt{2/3}\,\hbar\sqrt{2}\,|0,-1/2\rangle + \sqrt{1/3}\,\hbar\,|0,-1/2\rangle = \hbar\,|\overset{j}{1/2},\overset{m}{-1/2}\rangle$$

$$\sqrt{\frac{2}{3}}\,|-1,1/2\rangle + \left(-\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)|0,-1/2\rangle = |1/2,-1/2\rangle$$

$$\boxed{\sqrt{2/3}\,|-1,1/2\rangle - \sqrt{1/3}\,|0,-1/2\rangle = |1/2,-1/2\rangle}$$

* <u>otro método podría ser</u>:

$$J_-|1/2,-1/2\rangle = (L_-+S_-)\left(\sqrt{2/3}\,|-1,1/2\rangle - \sqrt{1/3}\,|0,-1/2\rangle\right) =$$

$$J_-|1/2,-1/2\rangle = -\sqrt{1/3}\,\hbar\sqrt{2}\,|-1,-1/2\rangle + \sqrt{2/3}\,\hbar\,|-1,-1/2\rangle = 0$$

$$|\overset{j}{1/2},\overset{m}{-1/2}\rangle = A\,|0,-1/2\rangle + B\,|-1,1/2\rangle$$

CL de los estados con $m_\ell+m_s=-1/2$

mirando en la tabla

$$\boxed{|1/2,-1/2\rangle = \sqrt{1/3}\,|0,-1/2\rangle - \sqrt{2/3}\,|-1,1/2\rangle}$$

e) L_z en $|1/2, 1/2\rangle \equiv |\alpha\rangle$

$$\langle 1/2, 1/2 | L_z | 1/2, 1/2\rangle \equiv \langle L_z\rangle_{|\alpha\rangle}$$

$$\langle L_z\rangle_{|\alpha\rangle} = \left(\sqrt{1/3}\,\langle 1, 1/2, 0, 1/2| - \sqrt{2/3}\,\langle 1, 1/2, 1, -1/2|\right) L_z$$

$$\left(\sqrt{1/3}\,|1, 1/2, \underset{m_\ell}{0}, \underset{m_s}{1/2}\rangle - \sqrt{2/3}\,|1, 1/2, 1, -1/2\rangle\right)$$

$$= \left(\sqrt{1/3}\,\langle 1, 1/2, 0, 1/2| - \sqrt{2/3}\,\langle 1, 1/2, 1, -1/2|\right)\left(\underbrace{\sqrt{1/3}\,\hbar .0\, |1, 1/2, 0, 1/2\rangle}_{=0} - \sqrt{2/3}\,\hbar\, 1\, |1, 1/2, 1, -1/2\rangle\right)$$

$$\boxed{\langle L_z\rangle_{|\alpha\rangle} = \hbar\,\frac{2}{3}}$$

Para S_z la cuenta es

$$\langle S_z\rangle_{|\alpha\rangle} = \left(\sqrt{1/3}\,\langle 1, 1/2, 0, 1/2| - \sqrt{2/3}\,\langle 1, 1/2, 1, -1/2|\right) S_z \left(\sqrt{1/3}\,|1, 1/2, 0, 1/2\rangle - \sqrt{2/3}\,|1, 1/2, 1, -1/2\rangle\right)$$

$$= \left(\sqrt{1/3}\,\langle 1, 1/2, 0, 1/2| - \sqrt{2/3}\,\langle 1, 1/2, 1, -1/2|\right)\left(\sqrt{1/3}\,\hbar\, 1/2\, |1, 1/2, 0, 1/2\rangle + \sqrt{2/3}\,\hbar\, 1/2\, |1, 1/2, 1, -1/2\rangle\right)$$

$$\langle S_z\rangle_{|\alpha\rangle} = 1/3\,\hbar\, 1/2 - 2/3\,\hbar\, 1/2 = \boxed{-\frac{\hbar}{6} = \langle S_z\rangle_{|\alpha\rangle}}$$

$$\langle J_z\rangle = \langle L_z\rangle + \langle S_z\rangle = \hbar\left(\frac{2}{3} - \frac{1}{6}\right) = \frac{\hbar}{2}$$

2.

$J_1 = 1 \quad J_2 = 1 \qquad J = 2, 1, 0 \qquad \{|J_1, J_2, m_1, m_2\rangle\}$ base ① usual

$J = J_1 + J_2 \qquad 0 = |J_1 - J_2| \leq J \leq J_1 + J_2 = 2 \qquad \{|J_1, J_2, J, m\rangle\}$ base ②

$m = m_1 + m_2 \qquad -J \leq m \leq J \qquad m = -2, -1, 0, 1, 2$

Los autoestados en la base ② son $\{|2,-2\rangle, |2,-1\rangle; |2,0\rangle; |2,1\rangle; |2,2\rangle; |1,-1\rangle; |1,0\rangle; |1,1\rangle; |0,0\rangle\}$

$m_1 = -1, 0, +1 \qquad J_1 = 1 \Rightarrow \quad + \equiv +1$ Expresaremos base ① $\{|m_1, m_2\rangle\}$

$m_2 = -1, 0, +1 \qquad J_2 = 1 \qquad - \equiv -1$

$0 \equiv 0$ base ② $\{|J, m\rangle\}$

$$\boxed{|J=2, m=2\rangle = |+, +\rangle} \qquad \boxed{|J=2, m=-2\rangle = |-, -\rangle}$$

los coef. de Clebsh-Gordan son $1 = \langle 1, 1 | 2, 2\rangle \qquad 1 = \langle -1, -1 | 2, -2\rangle$

Queré construir $|J=2, m=1\rangle$ a partir de $|J=2, m=2\rangle$ Como

$$|2,1\rangle = A\,|1,0\rangle + B\,|0,1\rangle \qquad A = \langle 1,0|2,1\rangle \;;\; B = \langle 0,1|2,1\rangle$$

oy un
lotudo

A,B son los únicos C-G que me interesarán ⇒ pruebo: $J=2,\ m=1$

upper)) $\sqrt{1.4}\,\langle 1,1|2,2\rangle = \sqrt{2}\,\langle 0,1|2,1\rangle + \sqrt{2}\,\langle 1,0|2,1\rangle$ $\quad m_1 = 1,\ m_2 = 1$

lower)) $\sqrt{4.1}\,\langle 1,0|2,1\rangle = \underbrace{0.\,\langle 2,0|2,2\rangle}_{=0} + \sqrt{2}\,\langle 1,1|2,2\rangle$

$J=2 \quad m=2 \qquad m_1 = 1,\ m_2 = 0$

luego como $\langle 1,1|2,2\rangle = \langle ++|2,2\rangle = 1 \Rightarrow$

$\langle +0|2,1\rangle = \frac{1}{\sqrt{2}} \quad \wedge \quad 1 = \frac{\langle 0+|2,1\rangle}{\sqrt{2}} + \frac{1}{2}$

$$\boxed{|J=2, m=1\rangle = \sqrt{1/2}\,|+0\rangle + \sqrt{1/2}\,|0+\rangle} \quad \Leftarrow \quad \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \langle 0+|2,1\rangle$$

Ahora $|2,0\rangle = A\,|0,0\rangle + B\,|-1,1\rangle + C\,|1,-1\rangle \Rightarrow$ necesitaré los coeficientes

$\langle 0,0|2,0\rangle$
$\langle -1,1|2,0\rangle$
$\langle 1,-1|2,0\rangle$

upper) $\sqrt{\cancel{2}.3}\,\langle 1,0|2,1\rangle = \sqrt{\cancel{2}}\,\langle 0,0|2,0\rangle + \sqrt{\cancel{2}}\,\langle 1,-1|2,0\rangle$

upper) $\sqrt{\cancel{2}.3}\,\langle 0,1|2,1\rangle = \sqrt{\cancel{2}}\,\langle -1,1|2,0\rangle + \sqrt{\cancel{2}}\,\langle 0,0|2,0\rangle$

lower) $\sqrt{\cancel{2}.3}\,\langle 0,0|2,0\rangle = \sqrt{\cancel{2}}\,\langle 1,0|2,1\rangle + \sqrt{\cancel{2}}\,\langle 0,1|2,1\rangle$

$J=2 \quad m=0$
$m_1=1 \quad m_2=0$

$J=2 \quad m=0$
$m_1=0 \quad m_2=1$

$J=2 \quad m=1$
$m_1=0 \quad m_2=0$

$\therefore \quad \langle 0\,0|2,0\rangle = \frac{2}{\sqrt{2}\sqrt{3}} = \sqrt{2/3}$

$\sqrt{3/2} = \langle -1,1|2,0\rangle + \sqrt{2/3} \Rightarrow \quad \langle -+|2,0\rangle = \frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}} = \frac{3-2}{\sqrt{6}} = \sqrt{1/6}$

$\langle +-|2,0\rangle = \sqrt{1/6}$

$$\boxed{|J=2, m=0\rangle = \sqrt{2/3}\,|0\,0\rangle + \sqrt{1/6}\,|-+\rangle + \sqrt{1/6}\,|+-\rangle}$$

$|2,-1\rangle = A\,|0,-1\rangle + B\,|-1,0\rangle \rightarrow$ necesitaré $\langle 0,-1|2,-1\rangle$
$\langle -1,0|2,-1\rangle$

upper) $\sqrt{3.2}\,\langle 1,-1|2,0\rangle = \sqrt{2}\,\langle 0,-1|2,-1\rangle + \underbrace{\sqrt{\ }}_{=0}\langle 1,\ |2,-1\rangle \quad J=2 \quad m=-1$
$m_1=1 \quad m_2=-1$

$\sqrt{\cancel{6}}\cdot\sqrt{1/\cancel{6}} = \sqrt{2}\,\langle 0,-1|2,-1\rangle$

$\frac{1}{\sqrt{2}} = \langle 0-|2,-1\rangle$

lower) $\overset{\sqrt{2}}{\sqrt{\cancel{4}}}\,\langle -1,-1|2,-2\rangle = \sqrt{\cancel{2}}\,\langle 0,-1|2,-1\rangle + \sqrt{\cancel{2}}\,\langle -1,0|2,-1\rangle \quad J=2 \quad m=-1$
$m_1=-1 \quad m_2=-1$

$\sqrt{2} = 1/\sqrt{2} + \langle -0|2,-1\rangle$

$\frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1/\sqrt{2} = \langle -0|2,-1\rangle$

$$\boxed{|J=2, m=-1\rangle = 1/\sqrt{2}\,|0-\rangle + 1/\sqrt{2}\,|-0\rangle}$$

Ahora $|1,-1\rangle = A\,|0\,-1\rangle + B\,|-1\,0\rangle \longrightarrow$ necesito $A = \langle 0,-1|1,-1\rangle$
$B = \langle -1,0|1,-1\rangle$

upper) $\sqrt{2}\,\langle 1,1|1,0\rangle = \underbrace{0.\langle -2,1|1,-1\rangle}_{=0} + \sqrt{2}\,\langle -1,0|1,-1\rangle \quad J=1 \quad m=-1$
$m_1=-1 \quad m_2=1$

lower) $\underbrace{0.\langle -1,-1|1,-2\rangle}_{=0} = \sqrt{2}\,\langle 0,-1|1,-1\rangle + \sqrt{2}\,\langle -1,0|1,-1\rangle \quad J=1 \quad m=-1$
$m_1=-1 \quad m_2=-1$

$$\left.\begin{array}{l} \langle -0|1,-1\rangle = -\langle 0-|1,-1\rangle \\ \langle -0|1,-1\rangle = \langle -+|1,0\rangle \end{array}\right\}[2]$$

Pero ahora necesito vincular estados de $J=2$ con $J=1$ para obtener uno de los productos escalares en [2]. Pero las relaciones de recurrencia no me sirven pues vinculan productos escalares (coeficientes con igual J). Aplicaremos ortonormalidad:

$A^2 + B^2 = 1 \qquad B = -A \quad \rightarrow \quad 2A^2 = 1 \qquad A = 1/\sqrt{2},\ B = -1/\sqrt{2}$

$$\boxed{|J=1, m=-1\rangle = 1/\sqrt{2}\,|0\,-\rangle - 1/\sqrt{2}\,|-\,0\rangle}$$

Ahora $|1,0\rangle = A|-1,1\rangle + B|1,-1\rangle + C|0,0\rangle$ → necesito

$A = \langle -1,1|1,0\rangle$
$B = \langle 1,-1|1,0\rangle$
$C = \langle 0,0|1,0\rangle$

Pero por lo hecho anteriormente tengo $A = \langle -+|1,0\rangle = -1/\sqrt{2}$

upper) $\sqrt{2}\langle 1,0|1,1\rangle = \sqrt{2}\,\langle 0,0|1,0\rangle + \sqrt{2}\,\langle 1,-1|1,0\rangle$ $J=1$ $m=0$ $m_1=1$ $m_2=0$

lower) $\sqrt{2}\langle -1,0|1,1\rangle = \sqrt{2}\,\langle 0,0|1,0\rangle + \sqrt{2}\,\langle -1,1|1,0\rangle$ $J=1$ $m=0$ $m_1=-1$ $m_2=0$

$$\cancel{\sqrt{2}}\cdot -\frac{1}{\cancel{\sqrt{2}}} = \sqrt{2}\,\langle 0,0|1,0\rangle - \cancel{\sqrt{2}}\cdot 1/\cancel{\sqrt{2}}$$

$$-1 = \sqrt{2}\,\langle 0,0|1,0\rangle - 1 \rightarrow C=0 \rightarrow B = 1/\sqrt{2}$$

$$\boxed{|J=1, m=0\rangle = -1/\sqrt{2}\,|-+\rangle + 1/\sqrt{2}\,|+-\rangle}$$

Ahora $|1,1\rangle = A|0,1\rangle + B|1,0\rangle$ → necesito $\langle 0,1|1,1\rangle = A$
$\langle 1,0|1,1\rangle = B$

, pero he calculado $\langle 1,-1|1,0\rangle = \langle 1,0|1,1\rangle = B = 1/\sqrt{2}$ ⇒

$$A^2 = 1 - B^2 = 1 - 1/2 = 1/2 \rightarrow |A| = 1/\sqrt{2}$$ faltaría ver el signo

lower) $\underbrace{\langle 0,0|1,0\rangle}_{=0} = \#_1\,\langle 1,0|1,1\rangle + \#_2\,\langle 0,1|1,1\rangle$ $J=1$ $m=1$ $m_1=0$ $m_2=0$

$$\rightarrow A = -1/\sqrt{2}$$

$$\boxed{|J=1, m=1\rangle = -1/\sqrt{2}\,|0\,+\rangle + 1/\sqrt{2}\,|+\,0\rangle}$$

Ahora $|0,0\rangle = A|-1,1\rangle + B|1,-1\rangle + C|0,0\rangle$ → necesitaré:

$A = \langle -1,1|0,0\rangle$
$B = \langle 1,-1|0,0\rangle$
$C = \langle 0,0|0,0\rangle$

upper) $0.\,\langle 1,0|0,1\rangle = \sqrt{2}\,\langle 0,0|0,0\rangle + \sqrt{2}\,\langle 1,-1|0,0\rangle$ $J=0$ $m=0$ $m_1=1$ $m_2=0$

no puede haber $J=0$ con $m\neq 0$

lower) $0.\,\langle -1,0|0,-1\rangle = \sqrt{2}\,\langle 0,0|0,0\rangle + \sqrt{2}\,\langle -1,1|0,0\rangle$ $J=0$ $m=0$ $m_1=-1$ $m_2=0$

$$\Rightarrow \langle 0,0|0,0\rangle = -\langle 1,-1|0,0\rangle$$
$$= -\langle -1,1|0,0\rangle$$
$$C = -A = -B \rightarrow A = B \rightarrow C^2 = A^2 = B^2$$

$$C^2 + A^2 + B^2 = 3A^2 = 1 \rightarrow |A| = 1/\sqrt{3} = |B| = |C|$$

$$\boxed{|J=0, m=0\rangle = 1/\sqrt{3}\,|-+\rangle + 1/\sqrt{3}\,|+-\rangle - 1/\sqrt{3}\,|0\,0\rangle}$$

3.

a)

base de autoestados de S^2, S_z (base $\{s,m\}$) $\ulcorner S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$

$\llcorner S_z|s,m\rangle = m\hbar\,|s,m\rangle$ $\lrcorner$

$$S^2 = (S_1+S_2)^2 = S_1^2 + S_2^2 + 2\,\vec{S}_1\cdot\vec{S}_2 \qquad (S^2 \text{ total})$$

$$S_z = S_z^1 + S_z^2 \qquad (S_z \text{ total})$$

- En la representación $\{m_1, m_2\}$ será

$|s_1, s_2, m_1, m_2\rangle \to \quad m_1 = +1/2, -1/2 \;;\; m_2 = +1/2, -1/2 \Rightarrow$

$|1/2, 1/2, 1/2, 1/2\rangle \equiv |++\rangle$ ①

$|1/2, 1/2, 1/2, -1/2\rangle \equiv |+-\rangle$ ②

$|1/2, 1/2, -1/2, 1/2\rangle \equiv |-+\rangle$ ③

$|1/2, 1/2, -1/2, -1/2\rangle \equiv |--\rangle$ ④

En ① es obvio que S_z será máxima, y en ④ mínima pues están alineados los spines, y en el mismo zentido $\Rightarrow$

$$|++\rangle = |\overset{s_1}{1/2}, \overset{s_2}{1/2}, s=1, m=1\rangle = \boxed{|\overset{s}{1}, \overset{m}{1}\rangle = |++\rangle}$$

$$|--\rangle = |1/2, 1/2, s=1, m=-1\rangle = \boxed{|\overset{s}{1}, \overset{m}{-1}\rangle = |--\rangle}$$

$$S_+ = S_x + i\,S_y \to \underset{\text{total}}{S_+} = S_+^1 + S_+^2 = S_x^1 + S_x^2 + i\left(S_y^1 + S_y^2\right) \Rightarrow$$

$$S_+|--\rangle = S_+|1,-1\rangle = \hbar\sqrt{s(s+1)-(-1)(-1+1)}\,|1,0\rangle = \hbar\sqrt{2}\,|1,0\rangle$$

$$= S_x^1|--\rangle + S_x^2|--\rangle + i\,S_y^1|--\rangle + i\,S_y^2|--\rangle$$

$$= \frac{\hbar}{2}|+-\rangle + \frac{\hbar}{2}|-+\rangle + i(-i)\frac{\hbar}{2}|+-\rangle + i(-i)\frac{\hbar}{2}|-+\rangle$$

$$= \hbar|+-\rangle + \hbar|-+\rangle \quad \to \quad \boxed{\frac{|+-\rangle + |-+\rangle}{\sqrt{2}} = |\overset{s}{1}, \overset{m}{0}\rangle}$$

$S_-|--\rangle = 0$ ∴ no me es de utilidad → planteo:

$$|0,0\rangle = A\,|+-\rangle + B\,|-+\rangle \Rightarrow \quad \langle 1,0|0,0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(\langle +-| + \langle -+|\right)\left(A|+-\rangle + B|-+\rangle\right) = \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} = 0 \Rightarrow A = -B$$

con $|A|^2 + |B|^2 = |A|^2 2 = 1 \to A = 1/\sqrt{2} \to \boxed{|0,0\rangle = 1/\sqrt{2}\left(|+-\rangle - |-+\rangle\right)}$

***NOTA**:

La base de autoestados de S^2, S_z es $\{|1,1\rangle; |1,-1\rangle; |-1,-1\rangle; |0,0\rangle\}$ donde

$-s \le m \le +s$ $\qquad m = 0, 1$

$\qquad s = -1, 0, +1$

b)

$$S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$

$$S^2 = S_{1x}^2 + S_{1y}^2 + \underbrace{S_{1z}^2}_{*} + S_{2x}^2 + S_{2y}^2 + \underbrace{S_{2z}^2}_{*} +$$

$$2S_{1x}S_{2x} + 2S_{1y} \cdot S_{2y} + 2\underbrace{S_{1z}S_{2z}}_{*}$$

Conozco los autoestados de *, pero no del resto ⇒ expresaré S^2 en términos de operadores que tengan a $\{m_1, m_2\}$ como autoestados.

matriz 4x4 en la base $\{m_1, m_2\}$

$$\{|++\rangle, |--\rangle, |+-\rangle, |-+\rangle\}$$

$$S_+ = \frac{S_x + iS_y}{\sqrt{2}}$$

$$S_- = \frac{S_x - iS_y}{\sqrt{2}}$$

$$S_+S_- = \frac{1}{2}\left(S_x^2 + iS_yS_x - iS_xS_y + S_y^2\right)$$

$$S_-S_+ = \frac{1}{2}\left(S_x^2 + iS_xS_y - iS_yS_x + S_y^2\right)$$

4. $$\hat{H} = \overbrace{\frac{p^2}{2m} - \frac{e^2}{r}}^{\equiv \hat{H}_0} + \frac{2\mu_B^2}{r^3}\frac{\vec{L}\cdot\vec{S}}{\hbar^2}$$; $\vec{S}$ ≡ spin del electrón

hamiltoniano para el átomo de hidrógeno en ausencia de campos externos. Pero un estado del átomo de hidrógeno se identifica:

$$\begin{array}{ll} & |n,\ell,m> \\ & \ \nearrow\ \nearrow\ \uparrow \\ \text{CCOC} \rightsquigarrow & H_0\ \ L^2\ \ L_z \end{array}$$ (sin spin)

$$\begin{array}{ll} & |n,\ell,m_\ell,m_s> \\ & \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow \\ \text{CCOC} \rightsquigarrow & H_0\ \ L^2\ \ L_z\ \ S_z \end{array}$$ (con spin) [sin acople spin-órbita]

[se mete el # $s=1/2$ correspondiente a $\hat{S}^2$]

$$\begin{array}{ll} & |n,\ell,\underbrace{\ell\pm\frac{1}{2}}_{\equiv j},m_j> \\ & \ \uparrow\ \ \swarrow\ \ \swarrow\ \ \searrow \\ \text{CCOC} \rightarrow & H\ \ L^2\ \ J^2\ \ J_z \end{array}$$ (con spin) [con acople spin-órbita]

a)

$$J^2 = (\vec{L}+\vec{S})^2 = L^2+S^2+2\vec{L}\cdot\vec{S} \Rightarrow \frac{1}{2}(J^2-L^2-S^2) = \vec{L}\cdot\vec{S}$$

- $$[H,L^2] = \underbrace{\left[\frac{p^2}{2m},L^2\right]}_{=0} - \underbrace{\left[\frac{e^2}{r},L^2\right]}_{=0} + \frac{\mu_B^2}{r^3\hbar^2}\left(\underbrace{[J^2,L^2]}_{=0} - \underbrace{[L^2,L^2]}_{=0} + \underbrace{[L^2,S^2]}_{=0}\right)$$

son parte del CCOC

pues $[\hat{L},\hat{S}]=0$ pues están en espacios diferentes

$$\Rightarrow \boxed{[H,L^2]=0}$$

- $$[H,S^2] = [H_0,S^2] + \frac{\mu_B^2}{r^2\hbar^2}\left([J^2,S^2] - \underbrace{[L^2,S^2]}_{=0} - \underbrace{[S^2,S^2]}_{=0}\right)$$

por $[\hat{L},\hat{S}]=0$ al estar en espacios diferentes

$$\vec{J}^2 = L^2+S^2+2\vec{L}\cdot\vec{S}$$
$$J^2 = L^2+S^2+2L_zS_z + L_+S_- + L_-S_+$$
$$[J^2,S^2] = \underbrace{[L^2,S^2]}_{=0} + \underbrace{[S^2,S^2]}_{=0} + 2\underbrace{[L_zS_z,S^2]}_{=0} + [L_+S_-,S^2] + [L_-S_+,S^2]$$

$$[L_+S_-,S^2] = -[S^2,L_+S_-] = -\left(L_+\overbrace{[S^2,S_-]}^{=0} + \overbrace{[S^2,L_+]}^{=0}S_-\right)$$ por estar en espacios diferentes

$$[L_-S_+,S^2] = -[S^2,L_-S_+] = -\left(L_-\underbrace{[S^2,S_+]}_{=0} + \underbrace{[S^2,L_-]}_{=0}S_+\right) \Rightarrow [J^2,S^2]=0$$ por estar en espacios diferentes

$$\Rightarrow \boxed{[H,S^2]=0}$$

- $$[H,J^2] = [H_0,J^2] + \frac{\mu_B^2}{r^3\hbar^2}\left(\underbrace{[J^2,J^2]}_{=0} - \underbrace{[L^2,J^2]}_{=0} - \underbrace{[S^2,J^2]}_{=0}\right)$$

son parte del CCOC

por lo anterior

$$\Rightarrow \boxed{[H,J^2]=0}$$

- $$[H, L_z] = \underbrace{[H_0, L_z]}_{=0} + \frac{\mu_B^2}{r^3\hbar^2}\left([J^2, L_z] - [L^2, L_z] - \underbrace{[S^2, L_z]}_{=0}\right)$$

están en espacios diferentes

$$[J^2, L_z] = [L^2, L_z] + \underbrace{[S^2, L_z]}_{=0} + 2\underbrace{[L_zS_z, L_z]}_{=0} + [L_+S_-, L_z] + [L_-S_+, L_z]$$

$$-[L_z, L_+S_-] = -(L_+[L_z, S_-] + [L_z, L_+]S_-)$$

$$= L_+\underbrace{[S_-, L_z]}_{=0} + \underbrace{[L_+, L_z]S_-}_{-\hbar L_+S_-} = -\hbar L_+S_-$$

$$-[L_z, L_-S_+] = -(L_-\underbrace{[S_+, L_z]}_{=0} + \underbrace{[L_z, L_-]S_+}_{-\hbar L_-S_+}) = +\hbar L_-S_+$$

$$-[L_z, L_zS_z] = -(L_z[L_z, S_z] + [L_z, L_z]S_z) = 0$$

$$[H, L_z] = \frac{\mu_B^2}{r^3\hbar^2}\hbar(L_-S_+ - L_+S_-)$$

$$\{(L_x - iL_y)(S_x + iS_y) - (L_x + iL_y)(S_x - iS_y)\}$$

$$\cancel{L_xS_x} - iL_yS_x + iL_xS_y + \cancel{L_yS_y} - \cancel{L_xS_x} - iL_yS_x + iL_xS_y - \cancel{L_yS_y}$$

$$\boxed{[H, L_z] = \frac{\mu_B^2 2i}{r^3\hbar}(L_yS_x + L_xS_y) = \frac{\mu_B^2}{r^3\hbar}(L_-S_+ - L_+S_-)}$$

- $$[H, S_z] = \underbrace{[H_0, S_z]}_{=0} + \frac{\mu_B^2}{r^3\hbar^2}\left([J^2, S_z] - \underbrace{[L^2, S_z]}_{=0} - [S^2, S_z]\right)$$

$=0$ → están en subespacios diferentes

$$[J^2, S_z] = \underbrace{[L^2, S_z]}_{=0} + [S^2, S_z] + 2\underbrace{[L_zS_z, S_z]}_{=0} + [L_+S_-, S_z] + [L_-S_+, S_z]$$

$$[J^2, S_z] - [S^2, S_z] = [L_+S_-, S_z] + [L_-S_+, S_z] = \hbar(L_+S_- - L_-S_+)$$

$$-[S_z, L_+S_-] = -(L_+\underbrace{[S_z, S_-]}_{-\hbar S_-} + \underbrace{[S_z, L_+]}_{=0}S_-) = \hbar L_+S_-$$

$$-[S_z, L_-S_+] = -(L_-\underbrace{[S_z, S_+]}_{\hbar S_+} + \underbrace{[S_z, L_-]}_{=0}S_+) = -\hbar L_-S_+$$

$$\boxed{[H, S_z] = \frac{\mu_B^2}{r^3\hbar}(L_+S_- - L_-S_+)}$$

- $$[H, J_z] = \underbrace{[H_0, J_z]}_{=0} + \frac{\mu_B^2}{r^3\hbar^2}\left(\underbrace{[J^2, J_z]}_{=0} - [L^2, J_z] - [S^2, J_z]\right)$$

$$[J^2, J_z] = 0 = [L^2, J_z] + [S^2, J_z] + \underbrace{2[L_zS_z, J_z]}_{\downarrow} + \underbrace{[L_+S_-, J_z] + [L_-S_+, J_z]}_{\downarrow}$$

$$2\underbrace{[L_zS_z, L_z]}_{=0} + 2\underbrace{[L_zS_z, S_z]}_{=0}$$

$$* \; [L_+S_-, L_z] + [L_+S_-, S_z] + [L_-S_+, L_z] + [L_-S_+, S_z]$$

$$\cancel{\hbar L_-S_+} - \cancel{\hbar L_+S_-} + \cancel{\hbar L_+S_-} - \cancel{\hbar L_-S_+} = 0$$

$$\Rightarrow [L^2, J_z] + [S^2, J_z] = 0$$

$\boxed{[H, J_z] = 0}$ como ya sabíamos por ser parte del CCOC b)

$$\hat{H} = \underbrace{\hat{H}_0 + \frac{2\mu_B^2}{r^3}\frac{\vec{L}\cdot\vec{S}}{\hbar^2}}_{\equiv \hat{H}_1} + \frac{\mu_B}{\hbar}B\left(\hat{L}_z + 2\hat{S}_z\right)$$

Ahora se enciende un campo magnético en $\hat{z}$ $\left(\vec{B}=B\hat{z}\right)$

- $[\hat{H},L^2] = \underbrace{[\hat{H}_1,\hat{L}^2]}_{=0} + \frac{\mu_B B}{\hbar}\underbrace{[L_z,L^2]}_{=0} + \frac{2\mu_B B}{\hbar}\underbrace{[S_z,L^2]}_{=0}$ → espacios diferentes

$$\Rightarrow \boxed{[\hat{H},L^2] = 0}$$

- $[\hat{H},S^2] = \underbrace{[H_1,S^2]}_{=0} + \frac{\mu_B B}{\hbar}\underbrace{[L_z,S^2]}_{=0} + \frac{2\mu_B B}{\hbar}\underbrace{[S_z,S^2]}_{=0}$

=0 → espacios diferentes

$$\Rightarrow \boxed{[\hat{H},\hat{S}^2] = 0}$$

- $[\hat{H},J^2] = \underbrace{[H_1,J^2]}_{=0} + \frac{\mu_B B}{\hbar}[L_z,J^2] + \frac{2\mu_B B}{\hbar}[S_z,J^2]$

$$[J^2,J_z] = 0 = [J^2,L_z] + [J^2,S_z]$$

$$[S_z,J^2] = [J^2,L_z]$$

$$\frac{\mu_B B}{\hbar}[L_z,J^2] - \frac{2\mu_B B}{\hbar}[L_z,J^2] = -\frac{\mu_B B}{\hbar}[L_z,J^2] = \frac{\mu_B B}{\hbar}[J^2,L_z]$$

$$= \frac{\mu_B B}{\hbar}\left(\hbar L_-S_+ - \hbar L_+S_-\right)$$

$$\Rightarrow \boxed{[\hat{H},J^2] = \mu_B B\left(L_-S_+ - L_+S_-\right)}$$

- $[H,L_z] = [H_1,L_z] + \frac{\mu_B B}{\hbar}\underbrace{[L_z,L_z]}_{=0} + \frac{2\mu_B B}{\hbar}\underbrace{[S_z,L_z]}_{=0}$

$$\Rightarrow \boxed{[H,L_z] = \frac{\mu_B^2}{r^3\hbar}\left(L_-S_+ - L_+S_-\right)}$$

no se altera este conmutador por el campo externo

- $[H,S_z] = [H_1,S_z] + \frac{\mu_B B}{\hbar}\underbrace{[L_z,S_z]}_{=0} + \frac{2\mu_B B}{\hbar}\underbrace{[S_z,S_z]}_{=0}$

$$\Rightarrow \boxed{[H,S_z] = \frac{\mu_B^2}{r^3\hbar}\left(L_+S_- - L_-S_+\right)}$$

- $[H,J_z] = [H_1,J_z] + \frac{\mu_B B}{\hbar}[L_z,J_z] + \frac{2\mu_B B}{\hbar}[S_z,J_z]$

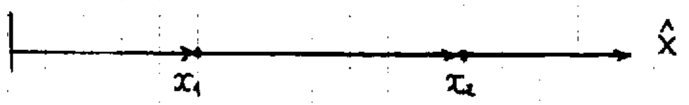
$$\frac{\mu_B B}{\hbar}\left(\underbrace{[L_z,L_z]}_{=0} + \underbrace{[L_z,S_z] + 2[S_z,L_z]}_{= \underbrace{[S_z,L_z]}_{=0}} + 2\underbrace{[S_z,S_z]}_{=0}\right)$$

$$\Rightarrow \boxed{[H,J_z] = 0}$$

Para el caso a) el CCOC sería: H, L^2, S^2, J^2, J_z

b) el CCOC sería: H, L^2, S^2, J_z (El campo $\vec{B}=B\hat{z}$ ha hecho perder la conmutatividad de J^2)

5.



$$U(x_1, x_2) = \frac{1}{2} m\omega^2(x_1-a)^2 + \frac{1}{2} m\omega^2(x_2+a)^2$$

a) $\vec{F}_i = -\overrightarrow{\text{grad}}_{x_i}(U) \Rightarrow \underset{\vec{F}_{x_1}}{-m\omega^2(x_1-a)} \quad \underset{\vec{F}_{x_2}}{-\, m\omega^2(x_2+a)}$

$\cancel{m}\ddot{x}_1 = -\cancel{m}\omega^2(x_1-a)$ $\qquad\qquad$ $\cancel{m}\ddot{x}_2 = -\cancel{m}\omega^2(x_2+a)$

$\ddot{x}_1 + \omega^2 x_1 = \omega^2 a$ $\quad$ Ecuaciones de Newton $\quad$ $\ddot{x}_2 + \omega^2 x_2 = -\omega^2 a$

$$H_1 = \frac{p_1^2}{2m} + \frac{m\omega^2}{2}(x_1-a)^2$$

6.

sistema $\quad j_1 = 1/2 \; , \; j_2 = 1/2 \quad |J_1 - J_2| = 0 \leq J \leq 1 = J_1 + J_2 \quad J = 0, 1$

$$\begin{tikzcd} & {|J,m\rangle} \arrow[dl] \arrow[dr] & \\ S^2 & & S_z \end{tikzcd} \qquad \begin{tikzcd} & {|m_1,m_2\rangle} \arrow[dl] \arrow[dr] & \\ S_{1z} & & S_{2z} \end{tikzcd}$$

} dos bases

$-J \leq m \leq J \rightarrow m = -1, 0, 1$

$m = m_1 + m_2$

Hay cuatro estados base

Medir es filtrar y por ende lterar)

$|J=1, m=-1\rangle$
$|J=1, m=0\rangle$
$|J=1, m=1\rangle$
$\underbrace{|J=0, m=0\rangle}$

$|m_1 = 1/2, m_2 = 1/2\rangle \equiv |++\rangle$
$|m_1 = 1/2, m_2 = -1/2\rangle \equiv |+-\rangle$
$|m_1 = -1/2, m_2 = 1/2\rangle \equiv |-+\rangle$
$|m_1 = -1/2, m_2 = -1/2\rangle \equiv |--\rangle$

singlete de spin

Sabemos que el sistema se halla en $\overline{S_{total} = 0} \Rightarrow$

$$|J=0, m=0\rangle = A\,|+-\rangle + B\,|-+\rangle \qquad A^2 + B^2 = 1$$

a) $$\left|S_{1z} = \frac{\hbar}{2}\right\rangle \otimes \mathbb{1}_2 = |+\,\square\rangle$$

$S_z = \hbar/2\rangle$
$= |+\rangle$

Si el observador B no realiza mediciones $m_2 = 1/2$ ó $m_2 = -1/2$; es decir que la partícula 2 no cambia de estado.
Revisando la tabla de Clebsh-Gordon resulta:

La otación de a ántica no es muy clara

$$|J=0, m=0\rangle = \underbrace{1/\sqrt{2}}_{\langle +-|J=0\; m=0\rangle}\,|+-\rangle \underbrace{-\, 1/\sqrt{2}}_{\langle -+|J=0\; m=0\rangle}\,|-+\rangle$$

La probabilidad de hallar el sistema en $|+-\rangle$ es igual a la de hallarlo en $|-+\rangle$
Pero $|J=0, m=0\rangle$ es una CL de autoestados de $S_{1z}, S_{2z} \Rightarrow$ al medir el observador A no "salgo" de $|J=0, m=0\rangle$.
Obtener $|S_{1z} = \hbar/2\rangle$ significará "Filtrar" el estado $|+-\rangle \Rightarrow$

$$\text{Prob}\left(S_{1z} = \hbar/2\right) = |\langle +-|J=0, m=0\rangle|^2 = \boxed{\frac{1}{2}}$$

Ahora veamos S_{1x}; $|J=0, m=0\rangle$ puede escribirse en CL de autoestados de S_{1x} y "leer" desde esa expresión la probabilidad.
Sea $$|S_x = \hbar/2\rangle = \frac{|+\rangle}{\sqrt{2}} + \frac{|-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|S_z = \hbar/2\rangle + |S_z = -\hbar/2\rangle\right)$$

$$|S_x = -\hbar/2\rangle = \frac{|+\rangle}{\sqrt{2}} - \frac{|-\rangle}{\sqrt{2}} \Rightarrow$$

$$\left.\begin{aligned} \sqrt{2}\,|-\rangle &= |S_x = \hbar/2\rangle - |S_x = -\hbar/2\rangle \\ \sqrt{2}\,|+\rangle &= |S_x = \hbar/2\rangle + |S_x = -\hbar/2\rangle \end{aligned}\right\} \Rightarrow$$

$$|J=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{\overset{\text{corresponde a la partícula 2}}{\left|S_x = \frac{\hbar}{2}, -\right\rangle}}\frac{1}{\sqrt{2}} + \left|S_x = -\frac{\hbar}{2}, -\right\rangle \frac{1}{\sqrt{2}}\right)$$

$$-\frac{1}{\sqrt{2}}\left(\underbrace{\left|S_x = \frac{\hbar}{2}, +\right\rangle}\frac{1}{\sqrt{2}} - \left|S_x = -\frac{\hbar}{2}, +\right\rangle \frac{1}{\sqrt{2}}\right)$$

Luego, la probabilidad serán los coef. de los términos al cuadrado sumados, entonces:

$$\text{Prob}\left(S_{1x} = \frac{\hbar}{2}\right) = \left(\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\,\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$$

b) La partícula 2 se halla en $S_{2z} = \hbar/2 \rightarrow S_{2z} = |\square +\rangle \Rightarrow$ mirando en:

$$|J=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$\Rightarrow \quad |J=0, m=0\rangle = |-+\rangle \quad \left(\begin{array}{l}\text{Porque la partícula 2 debe hallarse}\\ \text{con spin up}\end{array}\right)$$

i) Si el observador A mide S_{1z} medirá $-\frac{\hbar}{2}$ porque no la puede sacar del autoestado de S_z en el que se encuentra. Mide con probabilidad unidad (certeza).

ii) Si mide S_{1x} deberemos expresar $|-+\rangle$ en autoestados de S_{1x} para "leer" la probabilidad.

$$|J=0, m=0\rangle = |S_{1z} = -\frac{\hbar}{2}, S_{2z} = \frac{\hbar}{2}\rangle$$

$$= \left(|S_{1x} = \frac{\hbar}{2}, S_{2z} = \frac{\hbar}{2}\rangle - |S_{1x} = -\frac{\hbar}{2}, S_{2z} = \frac{\hbar}{2}\rangle\right)\frac{1}{\sqrt{2}}$$

Entonces, al medir el observador A podrá hallar a la partícula 1 en:

$$\begin{cases} S_{1x} = \frac{\hbar}{2} & \text{con prob. } \left(\frac{1}{2}\right) \\ S_{1x} = -\frac{\hbar}{2} & \text{con prob. } \left(\frac{1}{2}\right) \end{cases}$$

Pero en cualquiera de las dos opciones en las cuales caiga el observador A al medir $\left(+\frac{\hbar}{2} \text{ ó } -\frac{\hbar}{2}\right)$; si luego B mide S_{2z} obtendrá el mismo valor $+\frac{\hbar}{2}$, lo cual parece razonable por el hecho de que $[S_{1x}, S_{2z}] = 0$

7.

$$m\, |d^{(J)}_{mm'}(\beta)|^2 = m\, |\langle J, m| e^{-i\frac{J_y\beta}{\hbar}} |J, m'\rangle|^2$$

$$m\, |\langle J, m| \sum_{i=0}^{\infty} \left(\frac{-iJ_y\beta}{\hbar}\right)^n \frac{1}{n!} |J, m'\rangle|$$

$$|\sum_{n=0}^{\infty} m\, \langle J, m| \left(-\cancel{i}\left[\frac{J_+ - J_-}{2\cancel{i}\hbar}\right]\beta\right)^n \frac{1}{n!} |J, m'\rangle|^2$$

$$m\, \frac{1}{n!} \langle J, m| \left(1 - (J_+ - J_-)\frac{\beta}{2\hbar} + \left[(J_- - J_-)\frac{\beta}{2\hbar}\right]^2 - \dots\right) |J, m\rangle$$

$J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

$J_+ - J_- = iJ_y2$

$J_y = \frac{i}{2}(J_- - J$

$J_y = \frac{J_+ - J_-}{2i}$

8.

a) tensor esférico de rango 1 $\quad T_q^{(1)} = Y_1^q(\vec{W}) \quad -1 \leq q \leq 1$

con dos vectores $\vec{U}, \vec{V}$

$$|m| \leq \ell \qquad Y_\ell^m(\theta,\phi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}} \cdot e^{im\phi} \cdot \frac{1}{\operatorname{sen}^m\theta} \cdot \frac{d^{(\ell-m)}}{d(\cos\theta)^{\ell-m}}(\operatorname{sen}\theta)^{2\ell}$$

Los vectores $\vec{U}, \vec{V}$ cumplirán:

$$[U_i, J_j] = i\hbar\, \epsilon_{ijk} U_k \quad , \quad [V_i, J_j] = i\hbar\, \epsilon_{ijk} V_k \quad \longleftarrow$$ Cada vector es un tensor de rango 1

y necesitamos que

$$[J_z, T_q^1] = \hbar q\, T_q^1$$

$$[J_\pm, T_q^1] = \hbar\sqrt{(1\mp q)(1\pm q+1)}\; T^1_{q\pm 1} = \hbar\sqrt{(1\mp q)(2\pm q)}\; T^1_{q\pm 1}$$

Pero, con dos operadores vectoriales de rango 1 podemos formar un tensor de rango 1

$$[J_z, U_i] = -i\hbar\, \epsilon_{izk} U_k$$

$$[J_z, U_x] = -i\hbar(-1)U_y = i\hbar U_y$$
$$[J_z, U_y] = -i\hbar\, U_x$$
$$[J_z, U_z] = 0$$

* <u>Condiciones de Normalización</u>

$$U_0^{(1)} = \sqrt{\frac{3}{4\pi}}\, U_z \cdot N_0 \qquad \text{si defino } |U_z| \equiv 1 \rightarrow \quad N_0 = \sqrt{\frac{4\pi}{3}}$$

$$U_1^{(1)} = -\sqrt{\frac{3}{8\pi}} \cdot (U_x + iU_y) \cdot N_0$$

$$U_{-1}^{(1)} = \sqrt{\frac{3}{8\pi}} \cdot (U_x - iU_y) \cdot N$$

$$\rightarrow \begin{cases} U_0^{(1)} = U_z \\ U_1^{(1)} = -\frac{1}{\sqrt{2}}(U_x + iU_y) \\ U_{-1}^{(1)} = +\frac{1}{\sqrt{2}}(U_x - iU_y) \end{cases}$$

Puedo definir un tensor esférico de rango 1 partiendo de los armónicos esféricos, y luego usando dos tensores esféricos de rango 1 construirme otro de rango 0,1,2 mediante producto de los dos iniciales y un teorema que asevera:

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; k\, m \rangle\, X_{q_1}^{(k_1)}\, Y_{q_2}^{(k_2)}$$

$T_q^{(1)}$ con $|q| \leq 1$ $\quad T_{-1}^1, T_0^1, T_1^1$

$r.\cos\theta = z \equiv U_z$
$r.\text{sen}\theta. \cos\phi = x \equiv U_x$
$r.\text{sen}\theta. \text{sen}\phi = y \equiv U_y$

$$U_0^{(1)} \equiv Y_1^0(\vec{U}) \rightarrow \quad Y_1^0(\hat{n}) = \sqrt{\frac{3}{4\pi}} \cdot \frac{z}{r} \rightarrow \quad U_0^{(1)} = \sqrt{\frac{3}{4\pi}} U_z$$

$$Y_1^1(\hat{n}) = -\sqrt{\frac{3}{8\pi}} \left(\text{sen}\theta \cos\phi + i . \text{sen}\theta . \text{sen}\phi \right)$$

$$= -\sqrt{\frac{3}{8\pi}} \left(\frac{x}{r} + i \frac{y}{r} \right) \Rightarrow -\sqrt{\frac{3}{8\pi}} \left(U_x + i U_y \right) = U_1^{(1)}$$

$$Y_1^{-1}(\hat{n}) = +\sqrt{\frac{3}{8\pi}} \left(\frac{x}{r} - i \frac{y}{r} \right) \Rightarrow \quad U_{-1}^{(1)} = \sqrt{\frac{3}{8\pi}} \left(U_x - i U_y \right)$$

→ en forma análoga hallo $V_q^{(1)}$ $(|q| \leq 1) \Rightarrow$ $\qquad k_1 - k_2 \leq k \leq k_1 + k_2$

$$T_q^{(1)} = \sum_{q_1=-1}^{1} \sum_{q_2=-1}^{1} < 1,1; q_1, q_2 | 1,1; 1 q > U_{q_1}^{(1)} V_{q_2}^{(1)}$$

Coef de C-G con
$k_1 \rightarrow j_1 \quad k_2 \rightarrow j_2$
$1 = k \rightarrow j \quad q_{1,2} \rightarrow m_{1,2}$

$$T_0^{(1)} = \underbrace{<1,-1|1\,0>}_{\frac{1}{\sqrt{2}} . U_1^{(1)} V_{-1}^{(1)}} U_1^{(1)} V_{-1}^{(1)} + \underbrace{<-1,1|1\,0>}_{-\frac{1}{\sqrt{2}} U_{-1}^{(1)} V_1^{(1)}} U_{-1}^{(1)} V_1^{(1)} + \underbrace{<0,0|1,0>}_{+\ 0\ .\ U_z V_z} U_0^{(1)} V_0^{(1)}$$

$$T_0^{(1)} = -\frac{1}{2} \cdot \frac{1}{\sqrt{2}} \left(U_x + i U_y \right) \left(V_x - i V_y \right) + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \left(U_x - i U_y \right) \left(V_x + i V_y \right)$$

$$T_0^{(1)} = \frac{1}{\sqrt{2}\ 2} \left(\cancel{-U_x V_x} - i U_y V_x + i U_x V_y + \cancel{i^2 U_y V_y} + \cancel{U_x V_x} - i U_y V_x + i U_x V_y - \cancel{i^2 U_y V_y} \right)$$

$$\boxed{T_0^{(1)} = \frac{1}{\sqrt{2}} i . \left(U_x V_y - U_y V_x \right)}$$

$$T_1^{(1)} = < 1,0 | \overset{k}{1}\ \overset{q}{1} > U_1^{(1)} V_0^{(1)} + < 0,1 | \overset{k}{1}\ \overset{q}{1} > U_0^{(1)} V_1^{(1)}$$

$$= \frac{1}{\sqrt{2}} U_1^{(1)} V_0^{(1)} \quad - \frac{1}{\sqrt{2}} \quad U_0^{(1)} V_1^{(1)}$$

$$= \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}} \right) \quad \left(U_x + i U_y \right) V_z \quad - \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}} \right) \quad U_z \left(V_x + i V_y \right)$$

$$= -\frac{1}{2} \quad \left(U_x V_z + i U_y V_z \quad - U_z V_x \quad - i U_z V_y \right)$$

$$\boxed{T_1^{(1)} = -\frac{1}{2} \quad \left(U_x V_z - U_z V_x + i . \left[U_y V_z - U_z V_y \right] \right)}$$

$$T_{(-1)}^{(1)} = < -1,0 | 1\,-1 > U_{-1}^{(1)} V_0^{(1)} + < 0,-1 | 1,-1 > U_0^{(1)} V_{-1}^{(1)}$$

$$= -\frac{1}{\sqrt{2}} \sqrt{\frac{3}{8\pi}} . \sqrt{\frac{3}{4\pi}} \left(U_x - i U_y \right) V_z + \frac{1}{\sqrt{2}} \sqrt{\frac{3}{8\pi}} \sqrt{\frac{3}{4\pi}} \ U_z \left(V_x - i V_y \right)$$

$$\boxed{T_{(-1)}^{(1)} = -\frac{1}{2} \quad . \left(\left[U_x - i U_y \right] V_z + U_z \left[V_x - i V_y \right] \right)}$$

b) Para este caso procedemos de modo idem, utilizando la misma base hallada para $\vec{U}$ y $\vec{V}$ y el teorema:

$$T_q^{(k)} = \sum_{q_1, q_2} < k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k q > U_{q_1}^{k_1} V_{q_2}^{k_2}$$

, donde ahora querremos un tensor esférico de rango 2. Usamos

$$|k_1-k_2| \leq k \leq |k_1+k_2| \quad , \quad k_1 = 1 \; , k_2 = 1 \quad \Rightarrow \quad k = 0,1,\textcircled{2} \qquad k=2 \rightarrow$$

$$q = q_1+q_2 \qquad -k \leq q \leq k$$

$$q = -2,-1,0,1,2$$

aún 2 está en nuestras posibilidades ⇒

$$T_q^{(2)} = \sum_{q_1 q_2} < q_1 \, q_2 \, | 2 \; q > U_{q_1}^{(1)} V_{q_2}^{(1)} \qquad \therefore$$

$$U_0^{(1)} = U_z \qquad\qquad V_0^{(1)} = V_z$$

$$U_{\pm 1}^{(1)} = \mp\left(\frac{U_x \pm i U_y}{\sqrt{2}}\right) \qquad V_{\pm 1}^{(1)} = \mp\left(\frac{V_x \pm i V_y}{\sqrt{2}}\right)$$

Entonces:

$$T_1^{(2)} = \; <1 \;\, 0 \; | 2 \;\; 1 > U_1^{(1)} V_0^{(1)} + <0 \; 1 \; | 2 \; 1 > U_0^{(1)} V_1^{(1)}$$

$$\frac{1}{\sqrt{2}}\left(- \frac{U_x - i U_y}{\sqrt{2}}\right) V_z + \frac{1}{\sqrt{2}} \; U_z \left(\frac{-V_x - i V_y}{\sqrt{2}}\right)$$

$$\boxed{T_{+1}^{(2)} = \frac{1}{2}\left(- U_x V_z - i U_y V_z - U_z V_x - i U_z V_y\right)}$$

$$T_{-1}^{(2)} = \; <-1 \; 0 \; | 2 \; -1 > U_{-1}^{(1)} V_0^{(1)} + <0 \; -1 \; | \; 2 \; -1 > U_0^{(1)} V_{-1}^{(1)}$$

$$T_{-1}^{(2)} = \frac{1}{\sqrt{2}} \left(\frac{U_x - i U_y}{\sqrt{2}}\right) V_z + \frac{1}{\sqrt{2}} \, U_z \left(\frac{V_x - i V_y}{\sqrt{2}}\right)$$

$$\boxed{T_{-1}^{(2)} = \frac{1}{2} \left[U_x V_z - i U_y V_z + U_z V_x - i U_z V_y \right]}$$

$$T_0^{(2)} = \; <0 \; 0 \, | 2 \; 0 > U_0^{(1)} V_0^{(1)} + <1 \, -1 \, | \, 2 \, 0 > U_1^{(1)} V_1^{(1)} + <-1 \; 1 \, | \, 2 \, 0 > U_{-1}^{(1)} V_1^{(1)}$$

$$= \sqrt{\frac{2}{3}} \, U_z V_z + \frac{1}{\sqrt{6}} - \left(\frac{U_x + i U_y}{\sqrt{2}}\right) \left(\frac{V_x - i V_y}{\sqrt{2}}\right) + \frac{1}{\sqrt{6}} - \left(\frac{U_x - i U_y}{\sqrt{2}}\right) \left(\frac{V_x + i V_y}{\sqrt{2}}\right)$$

$$= \sqrt{2/3} \; U_z V_z - \sqrt{1/24} \, \Big(U_x V_x + i \cancel{U_y} V_x - i U_x V_y - i^2 U_y V_y$$

$$+ U_x V_x + i \cancel{U_x} V_y - i U_y V_x - i^2 U_y V_y \Big)$$

$$\boxed{T_0^{(2)} = \sqrt{2/3} \, U_z V_z - \sqrt{1/6} \; \left(U_x V_x + U_y V_y \right)}$$

$$\boxed{T_{+2}^{(2)} = \; <1 \;\, 1 \; | \, 2 \;\, 2 > U_1^{(1)} V_1^{(1)} = \frac{1}{2} \left(U_x + i U_y\right)\left(V_x + i V_y\right)}$$

$$\boxed{T_{-2}^{(2)} = \; <-1 \, -1 \; | \, 2 \;\, 2 > U_{-1}^{(1)} V_{-1}^{(1)} = \frac{1}{2} \left(U_x - i U_y\right)\left(V_x - i V_y\right)}$$

Podrían pedirnos aún el $T_0^{(0)}$ que sería:

$$T_0^{(0)} = \; <1 \, -1 \, | \, 0 \;\, 0 > U_1 V_{-1} + <-1 \; 1 \, | \, 0 \; 0 > U_{-1} V_1 + <0 \; 0 \, | \, 0 \, 0 > U_0 V_0$$

$$T_0^{(0)} = \sqrt{1/3} \; U_1^{(1)} V_{-1}^{(1)} + \sqrt{1/3} \; U_{-1}^{(1)} V_1^{(1)} - \sqrt{1/3} \; U_0^{(1)} V_0^{(1)}$$

9.

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm i V_y}{\sqrt{2}} \quad , \quad V_0^{(1)} = V_z$$

$$d^{(j=1)} = \begin{pmatrix} (1/2)(1+\cos\beta) & -(1/\sqrt{2})\,\mathrm{sen}\,\beta & (1/2)(1-\cos\beta) \\ (1/\sqrt{2})\,\mathrm{sen}\,\beta & \cos\beta & -(1/\sqrt{2})\,\mathrm{sen}\,\beta \\ (1/2)(1-\cos\beta) & (1/\sqrt{2})\,\mathrm{sen}\,\beta & (1/2)(1+\cos\beta) \end{pmatrix}$$

$\sum_{q'} d^{(1)}_{qq'}(\beta)\, V^{(1)}_{q'}$ ⟶ esto es un vector (V'_q) →

$$\begin{pmatrix} V'_{-1} \\ V'_0 \\ V'_1 \end{pmatrix} = \begin{pmatrix} d^{(j=1)}(\beta) \end{pmatrix} \begin{pmatrix} V_{-1} \\ V_0 \\ V_1 \end{pmatrix}$$

$V'_i = \sum_j R_{ij} V_j$ ← un vector transforma así

Para una rotación en torno a $\hat{y}$ la matriz $R_{\hat{y}}$ es:

$$R = \begin{pmatrix} \cos\alpha & 0 & \mathrm{sen}\,\alpha \\ 0 & 1 & 0 \\ -\mathrm{sen}\,\alpha & 0 & \cos\alpha \end{pmatrix} \rightarrow \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} \cos\alpha\, V_x + \mathrm{sen}\,\alpha\, V_z \\ V_y \\ -\mathrm{sen}\,\alpha\, V_x + \cos\alpha\, V_z \end{pmatrix}$$

∴ Por lo tanto un vector transforma de esta manera ante rotaciones en $\hat{y}$ ⟶
$$\begin{cases} V'_x = \cos\alpha\, V_x + \mathrm{sen}\,\alpha . V_z \\ V'_y = V_y \\ V'_z = -\mathrm{sen}\,\alpha . V_x + \cos\alpha\, V_z \end{cases}$$

$$\begin{pmatrix} V'_{-1} \\ V'_0 \\ V'_1 \end{pmatrix} = \begin{pmatrix} (1/2)(1+\cos\beta) V_{-1} - (1/\sqrt{2})\,\mathrm{sen}\,\beta\, V_0 + (1/2)(1-\cos\beta)\, V_1 \\ (1/\sqrt{2})\,\mathrm{sen}\,\beta\, V_{-1} \quad \cos\beta . V_0 \quad -(1/\sqrt{2})\,\mathrm{sen}\,\beta . V_1 \\ (1/2)(1-\cos\beta) V_{-1} \quad (1/\sqrt{2})\,\mathrm{sen}\,\beta . V_0 \quad (1/2)(1+\cos\beta)\, V_1 \end{pmatrix}$$

$$(V_{+1}) - (V_{-1}) = \left(-\frac{V_x - iV_y}{\sqrt{2}}\right) - \left(\frac{V_x - iV_y}{\sqrt{2}}\right) = -\frac{2V_x}{\sqrt{2}} = -\sqrt{2} . V_x$$

$$(V_{+1}) + (V_{-1}) = -\frac{V_x - iV_y}{\sqrt{2}} + \frac{V_x - iV_y}{\sqrt{2}} = -\frac{2iV_y}{\sqrt{2}} \rightarrow \quad V_x = \frac{V_{-1} - V_{+1}}{\sqrt{2}} \quad V_y = \frac{V_{-1} + V_{+1}}{-\sqrt{2}\, i}$$

Intuimos que $V'_y = V_y$ pues rotamos entorno a $\hat{y}$ ⇒

$$V'_y = \frac{V'_{-1} + V'_1}{-\sqrt{2}\, i} = -\frac{1}{\sqrt{2}\, i}\left(\cancel{2}\frac{1}{\cancel{2}} V_{-1} + \cancel{2}\frac{1}{\cancel{2}} V_1\right) = -\left(\frac{V_{-1} + V_1}{\sqrt{2}\, i}\right) = V_y \rightarrow \boxed{V'_y = V_y}$$

Así transforma un vector frente a rotaciones en el eje $\hat{y}$ (componente $\hat{y}$)

$$V'_x = \frac{V'_1 - V'_1}{\sqrt{2}} = \left(\cos\beta\, V_{-1} - \frac{2}{\sqrt{2}} \operatorname{sen}\beta\, V_0 - \cos\beta\, V_1\right)\frac{1}{\sqrt{2}}$$

$$= \frac{\cos\beta}{\sqrt{2}}\left(V_{-1} - V_1\right) - \operatorname{sen}\beta\, V_0$$

$$\boxed{V'_x = -\operatorname{sen}\beta\, V_z + \cos\beta\, V_x}$$

$$V'_z = V'_0 = \frac{1}{\sqrt{2}} \operatorname{sen}\beta \,.\, V_{-1} + \cos\beta \,.\, V_0 - \frac{1}{\sqrt{2}} \operatorname{sen}\beta \,.\, V_1$$

$$\boxed{V'_z = \operatorname{sen}\beta \,.\, V_x + \cos\beta \,.\, V_z}$$

Comparando con nuestra ecuación de transformación del vector vemos que $\alpha = -\beta$, es decir que dado nuestro sistema de coordenadas elegido (V_{-1}, V_0, V_{+1}) hemos representado una rotación en el sentido inverso al dado por α.

- **ACLARACIÓN POSTERIOR**

En realidad sucede esto porque la matriz $d^{(j=1)}$ está calculada con la siguiente elección de vectores:

$$d^{(j=1)} = \begin{array}{c} \\ \langle +1| \\ \langle 0| \\ \langle -1| \end{array} \begin{array}{c} |+1\rangle \quad |0\rangle \quad |-1\rangle \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array}\right) \end{array}$$

, y he utilizado otra que intercambia el $|+1\rangle$ con el $|-1\rangle$.

10.

Partícula sin spín ligada a un centro fijo mediante un potencial central.

$$H = \frac{p^2}{2m} - \frac{e}{r} \quad \rightarrow \text{su estado lo podemos dar con} \nearrow \quad |\underset{\underset{H}{\uparrow}}{n}, \underset{\underset{L^2}{\uparrow}}{l}, \underset{\underset{L_z}{\uparrow}}{m}\rangle$$

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}(x + iy) \equiv -U_{+1} \\ \\ z \equiv U_z \equiv U_0 \end{array}\right\} \text{Son parte de un operador tensorial} \quad U_q \doteq T^{(k=1)}_q$$

* Wigner-Eckart dice que.

$$\langle \alpha', j', m' | T^k_q | \alpha, j, m\rangle = \langle k j, m q | k j, j' m'\rangle \,.\, \frac{\langle \alpha' j' \| T^{(k)} \| \alpha j\rangle}{(2j+1)}$$

* Para un tensor de rango 1 $(k=1)$, un vector, se tiene:

$$\langle \alpha', j' m' | U_q | \alpha, j, m\rangle = \underbrace{\langle 1 j, m q | 1 j, j' m'\rangle}_{\substack{\text{el coeficiente de Clebsh-Gordan} \\ \text{de sumar momentos angulares} \\ 1 + \vec{J}\text{, con momentos en } \hat{z} \\ \text{dados por } m + q \;\rightarrow\; \text{valen} \\ m' = m + q \\ j - 1 \leq j' \leq j + 1 \qquad -j' \leq m' \leq j'}} \,.\, \frac{\langle \alpha' j' \| U' \| \alpha j\rangle}{(2j+1)}$$

$\langle n', l', m' | U_0 | n, l, m \rangle \propto \langle 1\, l ; m\, 0 | 1\, l , l' m' \rangle$ $\quad q=0 \rightarrow m' = m$ $\quad = m$ $\quad -j' = -l' \leq m' \leq l' = j'$

donde tomamos $l' \equiv j'$

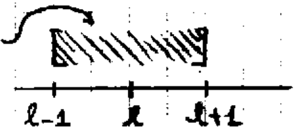
$l = j$

$-l' \leq m \leq l'$

* $l > 1 \rightarrow$ $\quad |l-1| = l-1 \leq l' \leq l+1$

$-1 \leq l'-l \leq +1$ $\rightarrow$ Solo pueden conectarse estados que difieren en una unidad de l

l' está acá



$l' = l-1 \rightarrow -l+1 \leq m \leq l-1$ $\quad 2(l-1)+1$ estados $\quad 2l-1$ estados

$l' = l \rightarrow -l \leq m \leq l \rightarrow$ $\quad 2l+1$ estados $\quad 2l+1$ stados

$l' = l+1 \rightarrow -l-1 \leq m \leq l+1$ $\quad 2(l+1)+1$ estados $\quad 2l+3$ estados

$$\therefore \boxed{\langle n', l', m' | U_0 | n, l, m \rangle \neq 0 \text{ si } \begin{array}{l} m = m' \\ |l-1| \leq l' \leq l+1 \end{array}}$$

* $l = 1 \rightarrow$ $\quad 0 \leq l' \leq 2$ $\quad l' = 0,1,2 \rightarrow$ $\quad m' = -2,-1,0,1,2$

$\langle n' l' m' | U_0 | n, 1, m \rangle \propto \langle 1\, 1 ; m\, 0 | 1\, 1 ; l'\, m \rangle$

Los no nulos serán los que tienen $l' = 0,1,2$ con $m' = -2,-1,0,1,2$

$\langle n' l' m' | U_{+1} | n\, l\, m \rangle \propto \langle 1\, l ; 1\, m | 1\, l , l' m' \rangle = \langle 1\, l ; 1, m | 1\, l ; l', 1+m \rangle$

$|l-1| \leq l' \leq l+1$ $\quad m' = 1+m$ $\leftarrow$ Sólo no nulo con

$-l' \leq m' \leq l'$

$-l' \leq 1+m \leq l'$

$$\boxed{\langle n' l' m' | U_{+1} | n\, l\, m \rangle \neq 0 \text{ si } \begin{array}{l} m' = m+1 \\ |l-1| \leq l' \leq l+1 \end{array}}$$

11.

a) $xy\ ,\ xz\ ,(x^2-y^2)$

$$X.Y = U_x.U_y \qquad\qquad U_z = U_0$$

$$= -\left(\frac{U_{-1}-U_1}{\sqrt{2}}\right)\left(\frac{U_{-1}+U_1}{\sqrt{2}\,i}\right) \qquad\qquad U_x = \frac{U_{-1}-U_1}{\sqrt{2}}$$

$$U_y = -\frac{(U_{-1}+U_1)}{\sqrt{2}\,i}$$

$$= -\frac{1}{2i}\Big(U_{-1}U_{-1} - U_1U_{-1} + U_{-1}U_1 - U_1U_1\Big)$$

$$x.y = -\frac{1}{2i}\left(U_{-1}^2 - U_1^2 + [U_{-1},U_1]\right) = -\frac{1}{2i}\left(U_{-1}^2 - U_1^2\right) = \frac{T_{+2}^{(2)} - T_{-2}^{(2)}}{2i}$$

Pero en un modo más sencillo es:

$$T_2^{(2)} = \frac{1}{2}(x+iy)(x+iy) \qquad T_{-2}^{(2)} = \frac{1}{2}(x-iy)(x-iy)$$

$$T_2^{(2)} = \frac{1}{2}(x^2-y^2+2ixy) \qquad T_{-2}^{(2)} = \frac{1}{2}(x^2-y^2-2ixy)$$

donde se ha tomado

$$\begin{cases} U_x = x = V_x \\ U_y = y = V_y \\ U_z = z = V_z \end{cases}$$

$$\boxed{xy = \frac{T_{+2}^{(2)} - T_{-2}^{(2)}}{2i}} \qquad \boxed{x^2-y^2 = T_{-2}^{(2)} + T_2^{(2)}}$$

$$T_0^{(2)} = \sqrt{\frac{1}{6}}\left(2U_zV_z - U_xV_x - U_yV_y\right)$$

$$T_0^{(2)} = \sqrt{\frac{1}{6}}\left(2z^2 - x^2 - y^2\right)$$

$$T_1^{(2)} = -\frac{1}{2}\left(x.z + iyz + z.x + izy\right) = -\left(xz + iyz\right) = -(x+iy)z$$

$$T_{-1}^{(2)} = \frac{1}{2}\left(xz - iyz + z.x - izy\right) = \left(xz - iyz\right) = (x-iy)z$$

$$\boxed{xz = \frac{T_{-1}^{(2)} - T_1^{(2)}}{2}}$$

b) $$Q = e<\alpha, J, m=J\,|\,(3z^2-r^2)\,|\,\alpha, J, m=J>$$

hay que evaluar:

$$e<\alpha, J, m'\,|\,(x^2-y^2)\,|\,\alpha, J, m=J> \equiv IV$$

hay que pasarlo a formato tensorial

$m' = J, J-1, J-2, \cdots)$

$$3z^2-r^2 = 2z^2 + z^2 - z^2 - x^2 - y^2 = 2z^2 - x^2 - y^2 \Rightarrow \qquad 1/\sqrt{6}\,(3z^2-r^2) = T_0^{(2)}$$

$$IV = e<\alpha, J, m'\,|\,T_{-2}^{(2)} + T_{\boxed{2}}^{(2)}\,|\,\alpha, J, \boxed{m=J}> = e<\ \ |T_{-2}^{(2)}|\ \ > + e<\ \ |T_{+2}^{(2)}|\ \ >$$

$$= e<2,J;-2,J\,|\,2,J;\ J, m'>C_j + e<2,J;2,J\,|\,2,J;J,m'>C_j$$

el coeficiente C_j (por Wigner-Eckart) No depende de m', m, q solo de J ⇒ dado que estamos en el mismo J será el mismo.

$$= e<-2, J\,|\,\underset{Ⓐ}{J, m'}>C_j + e<2, J\,|\,\underset{Ⓑ}{J, m'}>C_j$$

Usaremos las reglas de selección para C-G

$$\left.\begin{array}{c} m' = m+q \\ |J-2| \leq J' \leq J+2 \\ |m'| \leq J' \end{array}\right\} \Rightarrow$$

$$\Downarrow$$

Ⓐ $\langle -2, J \,|\, J, m' \rangle = \langle -2, J \,|\, J, J-2 \rangle$

un solo valor $\nearrow$ $\boxed{m' = J-2}$, $\quad J-2 \leq J \leq J+2$

$-2 \leq 0 \leq 2 \quad \forall\, J \geq 2$

$-J \leq m' \leq J \Rightarrow -J \leq J-2 \leq J$

$-2J \leq -2 \leq 0 \qquad \underline{\forall\, J \geq 1}$

$J<2$

$2-J \leq J \leq J+2$

$2 \leq 2J \leq 2J+2 \quad 1 \leq J < 2$

Ⓑ $\langle 2, J \,|\, J, m' \rangle = \langle 2, J \,|\, J, J+2 \rangle = 0$

$m' = J+2$

$-J \leq J+2 \leq J \Rightarrow -2J \leq 2 \leq 0 \rightarrow$ no vale

$$\mathbb{M} = e \langle -2, J \,|\, J, J-2 \rangle C_j = e \langle -2, J \,|\, J, J-2 \rangle \cdot \frac{\langle \alpha J \| T^{(2)} \| \alpha J \rangle}{(2J+1)}$$

Ahora utilicemos el dato propuesto:

$$Q = e \langle \alpha, J, J \,|\, \sqrt{6}\, T^{(2)}_{0} \,|\, \alpha, J, J \rangle$$

$$Q = e \underbrace{\langle 2, J; 0, J \,|\, 2, J; J, J \rangle}_{\equiv \langle 0, J | J, J \rangle} \cdot \frac{\langle \alpha J \| T^{2} \| \alpha J \rangle}{(2J+1)} \sqrt{6}$$

$$\boxed{\mathbb{M} = \frac{\langle 2J; -2, J \,|\, 2J; J, J-2 \rangle}{\langle 2J; 0, J \,|\, 2J; J, J \rangle} \, \frac{Q}{\sqrt{6}}}$$

, donde el valor de los coeficientes de Clebsh-Gordan se conocerá una vez determinado J