Practica 3

2.

$$U = \frac{a_0 + i\sigma.a}{a_0 - i\sigma.a} \qquad \sigma.a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a_3$$

a)

$$U^+ U = 1$$

$$U = \frac{a_0 + i\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}}{a_0 - i\begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + ia_2 & -a_3 \end{pmatrix}}$$

$$U^+ U = \left(\frac{\cancel{a_0 + i\sigma.a}}{\cancel{a_0 - i\sigma.a}}\right)\left(\frac{\cancel{a_0 - i\sigma.a}}{\cancel{a_0 + i\sigma.a}}\right)$$

$$U^+ = \frac{a_0 - i\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + a_2 i & -a_3 \end{pmatrix}}{a_0 + i\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}} \rightarrow (\sigma.a)^+ = \sigma.a$$

$$U^+ U = 1$$

$$\Rightarrow \boxed{U \text{ es unitaria}}$$

$$U = \frac{\begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + i\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}}{\begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} - i\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}} \longrightarrow \det = \{(a_0 + ia_3)(a_0 - ia_3) - (a_2 + ia_1)(-a_2 + ia_1)\}$$

$$a_0^2 + \cancel{ia_3 a_0} + a_3^2 - \cancel{ia_3 a_0} - (-a_2^2 - \cancel{ia_1 a_2} - a_1^2 + \cancel{ia_1 a_2})$$

$$a_0^2 + a_3^2 + a_2^2 + a_1^2$$

$$\det = \{(a_0 - ia_3)(a_0 + ia_3) - (-a_2 - ia_1)(+a_2 - ia_1)\} = \{a_0^2 - \cancel{ia_3 a_0} + \cancel{ia_3 a_0} + a_3^2 - (-a_2^2 - \cancel{ia_1 a_2} + \cancel{ia_1 a_2} - a_1^2\} = a_0^2 + a_3^2 + a_1^2 + a_2^2$$

$$\det(a_0 + i.\overleftrightarrow{\sigma}.\vec{a}) = \det(a_0 - i.\overleftrightarrow{\sigma}.\vec{a})$$

$$\Rightarrow$$

U tiene $\det = 1$

$\boxed{U \text{ es unimodular}}$

$$U = \frac{a_0 + i.\overleftrightarrow{\sigma}.\vec{a}}{a_0 - i\overleftrightarrow{\sigma}.\vec{a}} \cdot \frac{(a_0 + i\overleftrightarrow{\sigma}.\vec{a})}{a_0 + i\overleftrightarrow{\sigma}.\vec{a}} = \frac{a_0^2 + 2a_0 i\overleftrightarrow{\sigma}.\vec{a} - (\overleftrightarrow{\sigma}.\vec{a})^2}{a_0^2 + (\overleftrightarrow{\sigma}.\vec{a})^2}$$

$$= \frac{(a_0 + i\overleftrightarrow{\sigma}.\vec{a})^2}{a_0^2 + |\vec{a}|^2} = \frac{a_0^2 - |\vec{a}|^2 + 2a_0 i(\overleftrightarrow{\sigma}.\vec{a})}{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

$$U = \frac{(a_0^2 - |\vec{a}|^2)\mathbb{1} + i(\overleftrightarrow{\sigma}.\vec{a})2a_0}{(a_0^2 + |\vec{a}|^2)}$$

$$U = \underbrace{\frac{a_0^2 - a_1^2 - a_2^2 - a_3^2}{a_0^2 + a_1^2 + a_2^2 + a_3^2}}_{\cos\left(\frac{\phi}{2}\right)} \mathbb{1} + i\underbrace{\frac{(\overleftrightarrow{\sigma}.\vec{a})}{|\vec{a}|}}_{\overleftrightarrow{\sigma}.\hat{a}} \underbrace{\frac{2a_0|\vec{a}|}{a_0^2 + |\vec{a}|^2}}_{\text{sen}\left(\frac{\phi}{2}\right)}$$

$$\frac{\phi}{2} = \text{asen}\left(\frac{2a_0|\vec{a}|}{a_0^2 + |\vec{a}|^2}\right)$$

$$\left(\frac{a_0^2 - |\vec{a}|^2}{a_0^2 + |\vec{a}|^2}\right)^2 + \frac{(2a_0|\vec{a}|)^2}{(a_0^2 + |\vec{a}|)^2} = 1$$

$$(a_0^2 - |\vec{a}|^2)^2 + 4a_0^2|\vec{a}|^2 = (a_0^2 + |\vec{a}|^2)^2$$

$$\boxed{\hat{a} = \hat{n} = \frac{a_1 \hat{x}}{|\vec{a}|} + \frac{a_2}{|\vec{a}|}\hat{y} + \frac{a_3}{|\vec{a}|}\hat{z}}$$

$$\boxed{\frac{\phi}{2} = \text{asen}\left(\frac{2a_0|\vec{a}|}{a_0^2 + |\vec{a}|^2}\right) = \text{acos}\left(\frac{a_0^2 - |\vec{a}|^2}{a_0^2 + |\vec{a}|^2}\right)}$$

Esta matriz U representa una rotación $e^{-i\overleftrightarrow{\sigma}.\hat{a}.\frac{\phi}{2}}$

4.

Partícula de spin 1

$$|+\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |-\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad |0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\hat{A} \equiv S_z (S_z + \hbar)(S_z - \hbar)$$

$$S_z \begin{pmatrix}0\\1\\0\end{pmatrix} \cong J_z \begin{pmatrix}0\\1\\0\end{pmatrix} = \hbar \begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\left.\begin{aligned}(S_z - \hbar)|+\rangle &= 0\\ (S_z+\hbar)(2\hbar)|-\rangle &= 0\\ S_z(\hbar)(-\hbar)|0\rangle &= 0\end{aligned}\right\}$$

Todos los elementos son nulos $\boxed{(A)_{jk} = 0}$

$$J_x \begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{|+\rangle}{\sqrt{2}} + \frac{|-\rangle}{\sqrt{2}}$$

$$\hat{B} \equiv S_x (S_x + \hbar)(S_x - \hbar)$$

$\hat{B}|0\rangle$

$$-\frac{3}{\sqrt{2}}\frac{\hbar^3}{8}(|-\rangle + |+\rangle)$$

$\hat{B}|+\rangle$

$$(S_x+\hbar)\left(\frac{\hbar}{2}|-\rangle - \hbar|+\rangle\right)$$

$$S_x\left(-\frac{\hbar^2}{2}|-\rangle + \frac{\hbar^2}{4}|+\rangle + \frac{\hbar^2}{2}|-\rangle - \hbar^2|+\rangle\right) =$$

$$\cancel{-\frac{\hbar^3}{4}|+\rangle} + \frac{\hbar^3}{8}|-\rangle + \cancel{\frac{\hbar^3}{4}|+\rangle} - \frac{\hbar^3}{2}|-\rangle$$

$$-\frac{3\hbar^3}{8}|-\rangle$$

$\hat{B}|-\rangle$

$$(S_x+\hbar)\left(\frac{\hbar}{2}|+\rangle - \hbar|-\rangle\right)$$

$$S_x\left(\frac{\hbar^2}{4}|-\rangle + \frac{\hbar^2}{2}|+\rangle - \hbar^2|-\rangle - \frac{\hbar^2}{2}|+\rangle\right)$$

$$\frac{\hbar^3}{8}|+\rangle + \cancel{\frac{\hbar^3}{4}|-\rangle} - \frac{\hbar^3}{2}|+\rangle \cancel{+\frac{\hbar^3}{4}|-\rangle}$$

$$-\frac{3\hbar^3}{8}|+\rangle$$

$$B = \begin{array}{c|ccc} & |+\rangle & |-\rangle & |0\rangle \\ \hline \langle +| & -\frac{3\hbar^3}{8} & 0 & 0 \\ \langle -| & 0 & -\frac{3\hbar^3}{8} & 0 \\ \langle 0| & 0 & 0 & 0 \end{array}$$

$$\boxed{B = \frac{3\hbar^3}{8}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}}$$

5.

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right)$$ hamiltoniano de un cuerpo rígido

$$\frac{dK_1}{dt} = \frac{1}{i\hbar}[K_1, H] = \frac{1}{i\hbar}\left(\left[K_1, \frac{K_1^2}{2I_1}\right] + \left[K_1, \frac{K_2^2}{2I_2}\right] + \left[K_1, \frac{K_3^2}{2I_3}\right]\right)$$

$$= \frac{1}{i\hbar}\left(\overset{\shortparallel}{0}\ \frac{1}{2I_2}\left(K_2[K_1,K_2] + [K_1,K_2]K_2\right) + \frac{1}{2I_3}\left(K_3[K_1,K_3] + [K_1,K_3]K_3\right)\right)$$

$$\frac{dK_1}{dt} = \frac{1}{i\hbar}\left(\frac{1}{2I_2}[K_2\, i\hbar K_3 + i\hbar K_3 K_2] + \frac{1}{2I_3}[K_3 (-i\hbar) K_2 + -i\hbar K_2 K_3]\right)$$

$$\frac{dK_1}{dt} = \frac{1}{2I_2}(K_2K_3 + K_3K_2) - \frac{1}{2I_3}(K_3K_2 + K_2K_3) \Rightarrow$$ en forma ídem deberíamos llegar a.

Clásicamente $\longrightarrow$ $$\frac{dK_1}{dt} = \frac{K_2K_3}{I_2} - \frac{K_2K_3}{I_3} = K_2K_3\frac{(I_3 - I_2)}{I_2I_3}$$

$$\boxed{\frac{dK_1}{dt} = \frac{K_2K_3}{I_2I_3}(I_3 - I_2)}$$

$$\frac{dK_3}{dt} = \frac{1}{i\hbar}\left(\left[K_3, \frac{K_1^2}{2I_1}\right] + \left[K_3, \frac{K_2^2}{2I_2}\right]\right)$$

$$= \frac{1}{i\hbar}\left(\frac{1}{2I_1}\left(K_1[K_3,K_1] + [K_3,K_1]K_1\right) + \frac{1}{2I_2}\left(K_2[K_3,K_2] + [K_3,K_2]K_2\right)\right)$$

$$\frac{dK_3}{dt} = \left(\frac{K_1 \cancel{i\hbar} K_2 + \cancel{i\hbar} K_2 K_1}{\cancel{i\hbar}\, 2I_1} + \frac{-K_2 i\hbar K_1 - i\hbar K_1 K_2}{\cancel{i\hbar}\, 2I_2}\right)$$

clásicamente $\longrightarrow$ $$\boxed{\frac{dK_3}{dt} = \frac{K_1K_2}{I_1I_2}(I_2 - I_1)}$$

asimismo

$$\boxed{\frac{dK_2}{dt} = \frac{K_1K_3}{I_1I_3}(I_3 - I_1)}$$

7.

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \qquad \text{base } \{|+\rangle, |-\rangle\}$$

$$|\alpha\rangle = |+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle$$

$$\mathcal{D}(\hat{z},\varphi) = e^{-i\frac{S_z}{\hbar}\varphi} \rightarrow$$

$$|\alpha\rangle_R = e^{-iS_z\varphi/\hbar}|\alpha\rangle$$

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)$$

$$S_z|+\rangle = +\frac{\hbar}{2}|+\rangle$$

$$S_z|-\rangle = -\hbar/2\,|-\rangle$$

$$= \sum_{n=0}^{\infty}\left(\frac{-i\varphi}{\hbar}\right)^n S_z^n \left(|+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{-i\varphi}{\hbar}\right)^n\left(\frac{\hbar}{2}\right)^n\left(|+\rangle\langle +|\alpha\rangle\right) + \sum\left(\frac{i\varphi}{\hbar}\right)^n\left(\frac{\hbar}{2}\right)^n\left(|-\rangle\langle -|\alpha\rangle\right)$$

$$|\alpha\rangle_R = e^{-i\varphi/2}|+\rangle\langle +|\alpha\rangle + e^{i\varphi/2}|-\rangle\langle -|\alpha\rangle$$

estado arbitrario rotado

a)

$$_R\langle\alpha|S_x|\alpha\rangle_R = \langle S_x\rangle_R$$

$$= \left(\langle\alpha|+\rangle\langle +|e^{i\varphi/2} + \langle\alpha|-\rangle\langle -|e^{-i\varphi/2}\right)\frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)\left(e^{-i\varphi/2}|+\rangle\langle +|\alpha\rangle + e^{i\varphi/2}|-\rangle\langle -|\alpha\rangle\right)$$

$$= \left(\quad '' \quad\right)\frac{\hbar}{2}\left(|-\rangle\langle +|\alpha\rangle e^{-i\varphi/2} + |+\rangle\langle -|\alpha\rangle e^{i\varphi/2}\right)$$

$$= \frac{\hbar}{2}\left(\langle\alpha|+\rangle\langle -|\alpha\rangle e^{i\varphi} + \langle\alpha|-\rangle\langle +|\alpha\rangle e^{-i\varphi}\right)$$

$$= \frac{\hbar}{2}\left(\langle\alpha|+\rangle\langle -|\alpha\rangle + \langle\alpha|-\rangle\langle +|\alpha\rangle\right)\cos\varphi + i\cdot\left(\langle\alpha|+\rangle\langle -|\alpha\rangle - \langle\alpha|-\rangle\langle +|\alpha\rangle\right)\operatorname{sen}\varphi$$

$$\langle S_x\rangle_R = \frac{\hbar}{2}\left\{\langle\alpha|\left(|+\rangle\langle -| + |-\rangle\langle +|\right)|\alpha\rangle\right\}\cos\varphi + i\cdot\operatorname{sen}\varphi\left\{\langle\alpha|\left(|+\rangle\langle -| - |-\rangle\langle +|\right)|\alpha\rangle\right\}$$

$$\langle S_x\rangle_R = \langle\alpha|S_x|\alpha\rangle\cdot\cos\varphi - \langle\alpha|S_y|\alpha\rangle\cdot\operatorname{sen}\varphi$$

$$\boxed{\langle S_x\rangle_R = \langle S_x\rangle\cdot\cos\varphi - \langle S_y\rangle\operatorname{sen}\varphi}$$

Los valores de expectación transforman como vectores

b)

$$\mathcal{D}_{(\hat{z},2\pi)}|\alpha\rangle = -|+\rangle\langle +|\alpha\rangle - |-\rangle\langle -|\alpha\rangle = -|\alpha\rangle$$

$$\boxed{|\alpha\rangle_R = -|\alpha\rangle}$$

$$|\alpha\rangle_R = e^{i\pi}|\alpha\rangle$$

factor de fase

No se obtiene el mismo resultado sino el opuesto. Puede verse que si rotamos en 4π sí obtenemos el mismo resultado. (hay que dar dos vueltas)

8.

a)

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}(\vec{a}\times\vec{b})$$

$$(\sigma_1.a_1 + \sigma_2.a_2 + \sigma_3.a_3)(\sigma_1.b_1 + \sigma_2.b_2 + \sigma_3.b_3) =$$

$$\sum_j \sigma_j a_j \sum_k \sigma_k b_k = \sum_{\ell} a_\ell b_\ell + \sum_i i\sigma_i \overbrace{\varepsilon_{ijk} a_j b_k}^{(a\times b)_i}$$

$$\frac{1}{2} \underbrace{2i\sigma_i \varepsilon_{ijk}}_{[\sigma_j,\sigma_k]} = \frac{1}{2}(2i\sigma_i \varepsilon_{jki})$$

ijk
kij
jki

$$\sum_{j,k} a_j b_k \delta_{jk} + \sum_j \sum_k \frac{1}{2}[\sigma_j,\sigma_k] a_j b_k$$

$$= \sum_{jk} a_j b_k \left(\delta_{jk} + \frac{1}{2}[\sigma_j,\sigma_k]\right)$$

$$\sum_{j,k} \sigma_j a_j \sigma_k b_k = \sum_{j,k} a_j b_k \left(\frac{1}{2}\{\sigma_j,\sigma_k\} + \frac{1}{2}[\sigma_j,\sigma_k]\right)$$

$$\sum_{j,k} \sigma_j \sigma_k a_j b_k = \sum_{j,k} a_j b_k \left(\frac{\sigma_j\sigma_k + \cancel{\sigma_j\sigma_k} + \sigma_j\sigma_k - \cancel{\sigma_j\sigma_k}}{2}\right)$$

$$= \sum_{j,k} a_j b_k \cdot \sigma_j \cdot \sigma_k \Rightarrow \boxed{\text{vale la igualdad}}$$

b)

$$\mathcal{D}(\hat{n},\phi) = e^{-\left(\frac{i\vec{S}\cdot\hat{n}\phi}{\hbar}\right)}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}(\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3) \quad \text{con} \begin{cases} \vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3) \\ \hat{n} = (n_1,n_2,n_3) \end{cases} \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\mathcal{D}(\hat{n},\phi) = e^{-\left(\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)}$$

$$\mathcal{D}(\hat{n},\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{(-i)^{2n}(\vec{\sigma}\cdot\vec{n})^{2n}\phi^{2n}}{2^{2n}} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{(-i)^{2n+1}(\vec{\sigma}\cdot\vec{n})^{2n+1}\phi^{2n+1}}{2^{2n+1}}$$

$$\mathcal{D}(\hat{n},\phi) = \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!}\left(\frac{\phi}{2}\right)^{2n} + (-i)\vec{\sigma}\cdot\vec{n} \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n+1)!}\left(\frac{\phi}{2}\right)^{2n+1}$$

$$(-i)^{2n} = [(-i)^2]^n = (-1)^n \Rightarrow$$

$$(\vec{\sigma}\cdot\vec{n})^2 = |\vec{n}|^2 = 1$$
$$(\vec{\sigma}\cdot\vec{n})^4 = [(\vec{\sigma}\cdot\vec{n})^2]^2 = 1^2 = 1$$
$$(\vec{\sigma}\cdot\hat{n})^{2n} = [(\vec{\sigma}\cdot\vec{n})^2]^n = 1^n = 1 \quad \forall n \in N$$
$$(\vec{\sigma}\cdot\vec{n})^{2n+1} = (\vec{\sigma}\cdot\vec{n})^{2n}\,\vec{\sigma}\cdot\vec{n} = \vec{\sigma}\cdot\vec{n} \quad \forall n \in N$$

▲ usando la parte a)

$$\mathcal{D}(\hat{n},\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n (\phi/2)^{2n}}{(2n)!} - i\vec{\sigma}\cdot\vec{n} \sum_{n=0}^{\infty} \frac{(-1)^n (\phi/2)^{2n+1}}{(2n+1)!}$$

$$\boxed{\mathcal{D}(\hat{n},\phi) = \mathbb{1}\cos\left(\frac{\phi}{2}\right) - i\vec{\sigma}\cdot\vec{n}\,\operatorname{sen}\left(\frac{\phi}{2}\right)}$$

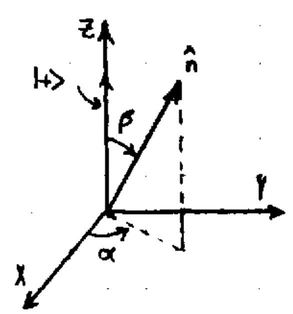
donde $\mathbb{1}$ es la matriz identidad c)

$$\mathcal{D}(\hat{n},\phi) \doteq \begin{pmatrix} \cos(\phi/2) & 0 \\ 0 & \cos(\phi/2) \end{pmatrix} - i \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix} \operatorname{sen}\left(\frac{\phi}{2}\right)$$

$$\boxed{\mathcal{D}(\hat{n},\phi) \doteq \begin{pmatrix} \cos(\phi/2) - i n_3 \operatorname{sen}(\phi/2) & (-i n_1 - n_2)\operatorname{sen}(\phi/2) \\ (-i n_1 + n_2)\operatorname{sen}(\phi/2) & \cos(\phi/2) + i n_3 \operatorname{sen}\left(\frac{\phi}{2}\right) \end{pmatrix}}$$

d)

$$\hat{n} = \hat{x}\cos\alpha \operatorname{sen}\beta + \hat{y}\operatorname{sen}\alpha \cdot \operatorname{sen}\beta + \hat{z}\cdot\cos\beta$$



$$|\vec{S}\cdot\hat{n}\,;+\rangle$$

$$\vec{S}\cdot\hat{n} = S_x\cdot\cos\alpha\cdot\operatorname{sen}\beta + S_y\cdot\operatorname{sen}\alpha\cdot\operatorname{sen}\beta + S_z\cdot\cos\beta$$

1. Hay que rotar β en torno a $\hat{y}$
2. Hay que rotar α en torno a $\hat{z}$

$$\mathcal{D}(\hat{y},\beta) = \mathbb{1}\cos(\beta/2) - i\,\sigma_y \cdot \operatorname{sen}(\beta/2)$$

$$\mathcal{D}(\hat{z},\alpha) = \mathbb{1}\cdot\cos(\alpha/2) - i\,\sigma_z\cdot\operatorname{sen}(\alpha/2)$$

$$|S\cdot\hat{n};+\rangle = \mathcal{D}(\hat{z},\alpha)\cdot\mathcal{D}(\hat{y},\beta)\,|+\rangle =$$

$$\begin{pmatrix} \cos(\alpha/2) - i\operatorname{sen}(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) + i\operatorname{sen}(\alpha/2) \end{pmatrix}\left[\begin{pmatrix} \cos(\beta/2) & 0 \\ 0 & \cos(\beta/2) \end{pmatrix} \overbrace{- i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}^{+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \operatorname{sen}(\beta/2)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|S\hat{n};+\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos(\beta/2) & -\operatorname{sen}(\beta/2) \\ \operatorname{sen}(\beta/2) & \cos(\beta/2) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Big(\qquad\Big)\begin{pmatrix} \cos(\beta/2) \\ \operatorname{sen}(\beta/2) \end{pmatrix} \Rightarrow \boxed{|S\cdot\hat{n},+\rangle = e^{-i\frac{\alpha}{2}}\cdot\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\frac{\alpha}{2}}\operatorname{sen}\left(\frac{\beta}{2}\right)|-\rangle}$$

9.

$$\mathcal{D}^{1/2}(\alpha,\beta,\gamma) = \mathcal{D}(\hat{z}\,\alpha)\cdot\mathcal{D}(\hat{y}\,\beta)\cdot\mathcal{D}(\hat{z}\,\gamma)$$

a)

$$\mathcal{D}^{1/2}(\alpha,\beta,\gamma) = e^{-i\frac{\vec{J}\cdot\hat{z}\,\alpha}{\hbar}}\, e^{-\frac{i\vec{J}\cdot\hat{y}\,\beta}{\hbar}}\, e^{-\frac{i\vec{J}\cdot\hat{z}\,\gamma}{\hbar}}$$

Pero $\longrightarrow$

$$\frac{\vec{J}\cdot\hat{n}}{\hbar} = \frac{J_{\hat{n}}}{\hbar} = \frac{S_{\hat{n}}}{\hbar} = \frac{\hbar\sigma_{\hat{n}}}{2\hbar}$$

$$\mathcal{D}^{1/2}(\alpha,\beta,\gamma) = e^{-i\frac{\sigma_z\alpha}{2}}\cdot e^{-\frac{i\sigma_y\beta}{2}}\cdot e^{-i\frac{\sigma_z\cdot\gamma}{2}}$$

$$\mathbb{1}\cdot\cos\left(\frac{\alpha}{2}\right) - i\,\sigma_z\cdot\operatorname{sen}\left(\frac{\alpha}{2}\right), \quad \sigma_z \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbb{1}\cos\left(\frac{\beta}{2}\right) - i\,\sigma_y\cdot\operatorname{sen}\left(\frac{\beta}{2}\right), \quad \sigma_y \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbb{1}\cdot\cos\left(\frac{\gamma}{2}\right) - i\,\sigma_z\cdot\operatorname{sen}\left(\frac{\gamma}{2}\right), \quad \sigma_z \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

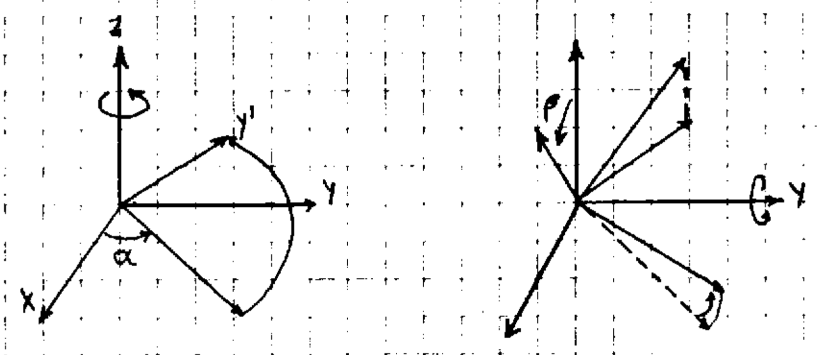
$$\begin{pmatrix}\cos\left(\frac{\alpha}{2}\right)-i\,sen\left(\frac{\alpha}{2}\right) & 0\\ 0 & \cos\left(\frac{\alpha}{2}\right)+i\,sen\left(\frac{\alpha}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\beta}{2}\right) & -sen\left(\frac{\beta}{2}\right)\\ sen\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\gamma}{2}\right)-i\,sen\left(\frac{\gamma}{2}\right) & 0\\ 0 & \cos\left(\frac{\gamma}{2}\right)+i\,sen\left(\frac{\gamma}{2}\right)\end{pmatrix}$$

$$\begin{pmatrix}e^{-i\alpha/2} & 0\\ 0 & e^{i\alpha/2}\end{pmatrix}\begin{pmatrix}\cos(\beta/2) & -sen(\beta/2)\\ sen(\beta/2) & \cos(\beta/2)\end{pmatrix}\begin{pmatrix}e^{-i\gamma/2} & 0\\ 0 & e^{i\gamma/2}\end{pmatrix}$$

$$\begin{pmatrix}e^{-i\frac{\alpha}{2}}\cdot\cos(\beta/2) & -e^{-i\alpha/2}\,sen(\beta/2)\\ e^{i\alpha/2}\cdot sen(\beta/2) & e^{i\alpha/2}\cdot\cos(\beta/2)\end{pmatrix}\begin{pmatrix}e^{-i\gamma/2} & 0\\ 0 & e^{i\gamma/2}\end{pmatrix}$$

$$\boxed{\mathcal{D}^{1/2}_{(\alpha,\beta,\gamma)}=\begin{pmatrix}e^{-i\frac{\alpha}{2}-i\frac{\gamma}{2}}\cdot\cos\left(\frac{\beta}{2}\right) & -e^{-i\frac{\alpha}{2}+i\frac{\gamma}{2}}\cdot sen\left(\frac{\beta}{2}\right)\\ e^{i\frac{\alpha}{2}-i\frac{\gamma}{2}}\cdot sen\left(\frac{\beta}{2}\right) & e^{i\frac{\alpha}{2}+i\frac{\gamma}{2}}\cdot\cos\left(\frac{\beta}{2}\right)\end{pmatrix}}$$

b)



Usando lo hecho en ejercicio 8c tenemos:

$$\mathcal{D}^{1/2}_{(\hat{n},\theta)}=\begin{pmatrix}\cos(\theta/2)-i\,n_3\,sen(\theta/2) & (-i\,n_1-n_2)\,sen(\theta/2)\\ (-i\,n_1+n_2)\,sen(\theta/2) & \cos(\theta/2)+i\,n_3\cdot sen(\theta/2)\end{pmatrix}$$

Le podemos sacar fase global fuera →

$$\mathcal{D}^{1/2}=e^{-i\alpha/2}\begin{pmatrix}1 & 0\\ 0 & e^{i\alpha}\end{pmatrix}\cdot\left(\,''\,\right)\begin{pmatrix}1 & \\ & e^{i\gamma}\end{pmatrix}e^{-i\gamma/2};\quad \mathcal{D}^{1/2}=e^{-i\alpha/2-i\gamma/2}\begin{pmatrix}\cos\left(\frac{\beta}{2}\right) & -e^{i\gamma}\cdot sen\left(\frac{\beta}{2}\right)\\ e^{i\alpha}\cdot sen\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)}\cdot\cos\left(\frac{\beta}{2}\right)\end{pmatrix}$$

$$\left(\cos^2\left(\theta/2\right)+n_3^2\cdot sen^2\left(\frac{\theta}{2}\right)\right)+\left(sen^2(\theta/2)\right)\left(n_1^2+n_2^2\right)=n_1^2+n_2^2+n_3^2=1$$

$$\left[\begin{array}{l}\cos\left(\frac{\theta}{2}\right)=\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\\ \cancel{i}\,n_3\cdot sen\left(\frac{\theta}{2}\right)=\cancel{i}\cos\left(\frac{\beta}{2}\right)\,sen\left(\frac{\alpha+\gamma}{2}\right)\\ \cancel{i}\,n_2\cdot sen\left(\frac{\theta}{2}\right)=\cancel{i}\,sen\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\gamma}{2}\right)\\ n_1\cdot sen\left(\frac{\theta}{2}\right)=sen\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\gamma}{2}\right)\end{array}\right.$$

datos α,β,γ

incógnitas θ, n_1, n_2, n_3

$$1=\cos^2\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha+\gamma}{2}\right)+\frac{1}{n_3^2}\cdot\cos^2\left(\frac{\beta}{2}\right)\cdot sen^2\left(\frac{\alpha+\gamma}{2}\right)$$

$$\boxed{\frac{1}{n_3^2} = \frac{1 - \cos^2(\beta/2)\cdot\cos^2(\alpha+\gamma/2)}{\cos^2(\beta/2)\cdot \operatorname{sen}^2(\alpha+\gamma/2)}}$$

$$1 = \cos^2\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha+\gamma}{2}\right) + \frac{1}{n_2^2}\cdot \operatorname{sen}^2\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha-\gamma}{2}\right)$$

$$\boxed{\frac{1}{n_2^2} = \frac{1 - \cos^2(\beta/2)\cdot\cos^2(\alpha+\gamma/2)}{\operatorname{sen}^2(\beta/2)\cdot\cos^2(\alpha-\gamma/2)}}$$

$$1 = \cos^2\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha+\gamma}{2}\right) + \frac{1}{n_1^2}\cdot \operatorname{sen}^2\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha-\gamma}{2}\right)$$

$$\boxed{\frac{1}{n_1^2} = \frac{1 - \cos^2(\beta/2)\cdot\cos^2(\alpha+\gamma/2)}{\operatorname{sen}^2(\beta/2)\cdot\cos^2(\alpha-\gamma/2)}}$$

$$n_1^2 = n_2^2 \rightarrow n_1 = n_2 \rightarrow \qquad n_3^2 = 2n_1^2 \rightarrow n_3 = \sqrt{2}\, n_1$$

$$\frac{\operatorname{sen}(\theta)}{2} = \operatorname{sen}\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\gamma}{2}\right)\frac{1}{n_2}\cdot\operatorname{sen}\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\gamma}{2}\right)$$

$$\frac{\operatorname{sen}(\theta)}{\cancel{2}} = \frac{\operatorname{sen}(\beta)}{\cancel{2}}\cdot\frac{\sqrt{1-\cos^2(\beta/2)\cdot\cos^2(\alpha+\gamma/2)}}{\operatorname{sen}(\beta/2)\cdot\cancel{\cos(\alpha-\gamma/2)}}\cdot\cancel{\cos(\alpha-\gamma/2)}\cdot\cos(\alpha+\gamma/2)$$

$$\sqrt{\operatorname{sen}^2(\beta/2) + \cos^2(\beta/2)\left[1 - \cos^2(\alpha+\gamma/2)\right]}$$

$$\operatorname{sen}(\theta) = \frac{\operatorname{sen}(\beta)}{\operatorname{sen}(\beta/2)}\cdot\sqrt{\operatorname{sen}^2\left(\frac{\beta}{2}\right) + \cos^2\left(\frac{\beta}{2}\right)\cdot\operatorname{sen}^2\left(\frac{\alpha+\gamma}{2}\right)}$$

$$\cos(\theta) = \cos\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$\operatorname{sen}\left(\frac{\beta}{2}+\frac{\beta}{2}\right) = 2\operatorname{sen}\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)$$

$$\operatorname{sen}\theta = 2\cdot\cos(\beta/2)\cdot\sqrt{\operatorname{sen}^2\left(\frac{\beta}{2}\right)+\cos^2\left(\frac{\beta}{2}\right)\cdot\operatorname{sen}^2\left(\frac{\alpha+\gamma}{2}\right)}$$

$$\cos\theta = \cos\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$\operatorname{sen}\theta = \frac{2\cdot\cos\theta}{\cos\left(\frac{\alpha+\gamma}{2}\right)}\cdot\sqrt{1-\cos^2(\beta/2)\cdot\cos^2(\alpha+\gamma/2)}$$

$$\tan\theta = 2\sqrt{\frac{1}{\cos^2\left(\frac{\alpha+\gamma}{2}\right)} - \cos^2(\beta/2)}$$

$$\boxed{\theta = \operatorname{atan}\left\{2^{1/2}\left(\cos^{-2}\left[\frac{\alpha+\gamma}{2}\right] - \cos^2\left[\frac{\beta}{2}\right]\right)^{1/2}\right\}}$$

13. $$[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k$$

a)
$$J_+ = J_x + iJ_y$$
$$J_- = J_x - iJ_y$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = [J_x + iJ_y, J_x] - [J_x + iJ_y, +iJ_y]$$
$$= \underbrace{[J_x, J_x]}_{0} + \underbrace{i[J_y, J_x]}_{-ii\hbar J_z} \underbrace{- i[J_x, J_y]}_{-i(i\hbar J_z)} + \underbrace{[J_y, J_y]}_{0}$$

$$\boxed{[J_+, J_-] = 2\hbar J_z}$$

$$[J_z, J_+] = [J_z, J_x] + i[J_z, J_y] = i\hbar J_y + i(-i\hbar J_x) = \hbar(J_x + iJ_y)$$
$$\boxed{[J_z, J_+] = \hbar J_+}$$

$$[J_z, J_-] = [J_z, J_x] - i[J_z, J_y] = i\hbar J_y - i(-i\hbar J_x) = -\hbar(J_x - iJ_y)$$
$$\boxed{[J_z, J_-] = -\hbar J_-}$$

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_xJ_x + iJ_yJ_x - iJ_xJ_y - i^2J_yJ_y$$
$$= J_x^2 + J_y^2 + i(J_yJ_x - J_xJ_y) = J^2 - J_z^2 + i(-i\hbar J_z)$$
$$\boxed{J_+J_- = J^2 - J_z^2 + \hbar J_z}$$

$$J_+J_-|j,m\rangle = J^2|j,m\rangle - J_z^2|j,m\rangle + \hbar J_z|j,m\rangle$$
$$J_+J_-|j,m\rangle = j(j+1)\hbar^2|j,m\rangle - m^2\hbar^2|j,m\rangle + \hbar^2 m|j,m\rangle$$
$$J_-J_+|j,m\rangle = ((J_+J_-) - [J_+, J_-])|j,m\rangle$$
$$= j(j+1)\hbar^2|j,m\rangle - m^2\hbar^2|j,m\rangle + \hbar^2 m|j,m\rangle - 2\hbar J_z|j,m\rangle$$
$$-2\hbar^2 m|j,m\rangle$$
$$J_-J_+|j,m\rangle = j(j+1)\hbar^2|j,m\rangle - m^2\hbar^2|j,m\rangle - \hbar^2 m|j,m\rangle$$
$$J_-J_+|j,m\rangle = \hbar^2\left(j(j+1) - m(m+1)\right)|j,m\rangle \Rightarrow J_- = J_+^\dagger$$
$$J_+J_-|j,m\rangle = \hbar^2\left(j(j+1) - m(m-1)\right)|j,m\rangle \qquad \boxed{J_+^\dagger J_+ \equiv J_+^2}$$

lo usaremos.

$$J_zJ_+|j,m\rangle = [J_z, J_+]|j,m\rangle + J_+J_z|j,m\rangle = \hbar J_+|j,m\rangle + J_+ m\hbar|j,m\rangle$$
$$J_z(J_+|j,m\rangle) = \hbar(m+1)(J_+|j,m\rangle)$$

$J_+|j,m\rangle$ es autoket de J_z con $\hbar(m+1)$ de autovalor
$|j,m+1\rangle$ es " " " " " " →

$$\Rightarrow \quad c_+|j,m+1\rangle = J_+|j,m\rangle$$

$$J_+ |j,m\rangle = c_+ |j,m+1\rangle \longrightarrow \langle j,m|(J_+)^\dagger = \langle j,m+1| c_+^*$$

$$(\langle j,m| \underbrace{J_-}_{(J_+)^\dagger} J_+ |j,m\rangle = c_+^* c_+ \langle j,m+1|j,m+1\rangle$$

$$\langle j,m| J_- J_+ |j,m\rangle = \hbar^2 (j[j+1] - m[m+1]) \langle j,m|j,m\rangle$$

$$\Rightarrow |c_+|^2 = \hbar^2 (j[j+1] - m[m+1])$$

Procediendo de modo ídem con $J_- J_- = (J_+)^\dagger J_-$

$$\boxed{J_\pm |j,m\rangle = \hbar \left(j[j+1] - m[m\pm 1]\right)^{1/2} |j,m\pm 1\rangle}$$

$$J_+ |j,+j\rangle = \hbar \underbrace{\left(j(j+1) - j(j+1)\right)}_{=0}^{1/2} |j,j+1\rangle \rightarrow \boxed{J_+ |j,+j\rangle = 0}$$

En forma ídem será $\boxed{J_- |j,-j\rangle = 0}$

b)

$$J_z |j,m\rangle = m\hbar |j,m\rangle$$

$$J_+ = J_x + iJ_y$$
$$J_- = J_x - iJ_y$$

$\langle \alpha | J_x | \alpha \rangle$, con $|\alpha\rangle$ autoestado de J_z

$$J_z|\alpha\rangle = \alpha|\alpha\rangle$$

$$\frac{J_+ + J_-}{2} = J_x$$

$$\frac{J_+ - J_-}{2i} = J_y$$

$$\langle J_x \rangle = \langle j,m| \frac{J_+ + J_-}{2} |j,m\rangle$$

$$\langle J_x \rangle = \frac{1}{2} \cdot \langle j,m| \left(|j,m+1\rangle + |j,m-1\rangle \right)$$

$$\boxed{\langle J_x \rangle = 0}$$

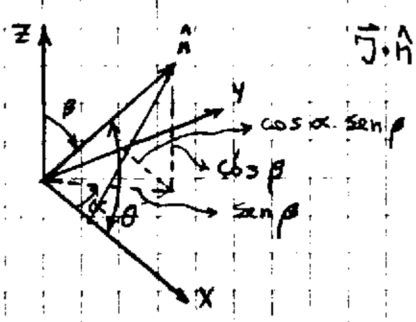
$$\langle J_y \rangle = \langle j,m| \frac{J_+ - J_-}{2i} |j,m\rangle$$

$$= \frac{1}{2i} \langle j,m| \left(|j,m+1\rangle - |j,m-1\rangle \right)$$

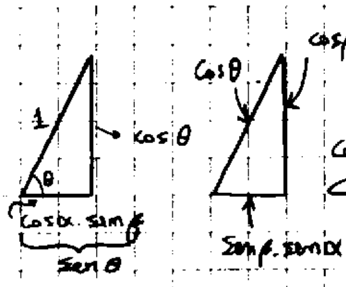
$$\boxed{\langle J_y \rangle = 0}$$

En $|\psi\rangle$ con $J_z|\psi\rangle = \hbar m |\psi\rangle$

$$\langle \psi | \vec{J}\cdot\hat{n} | \psi \rangle =$$



$$\vec{J}\cdot\hat{n} = J_x \cdot \cos\alpha \cdot \text{sen}\beta + J_y \cdot \text{sen}\alpha\, \text{sen}\beta + J_z \cdot \cos\beta$$



$$\text{sen}^2\beta + \cos^2\beta = 1$$

$$\cos^2\theta + \cos^2\alpha \cdot \text{sen}^2\beta = 1$$

$$\cos^2\theta + \cos^2\alpha (1 - \cos^2\beta) = 1$$

$$\cos^2\theta = \cos^2\beta + \text{sen}^2\beta \cdot \text{sen}^2\alpha$$

$$\langle \vec{J}\cdot\hat{n} \rangle_\psi = \cos\alpha \cdot \text{sen}\beta \left\langle \frac{J_+ + J_-}{2} \right\rangle + \text{sen}\alpha \cdot \text{sen}\beta \left\langle \frac{J_+ - J_-}{2i} \right\rangle + \langle J_z \rangle \cos\beta$$

$$\langle \vec{J}\cdot\hat{n}\rangle_\psi = \cos\alpha\,.\,\operatorname{sen}\beta\,\underbrace{\langle J_x\rangle_\psi}_{=0} + \operatorname{sen}\alpha\,.\,\operatorname{sen}\beta\,\underbrace{\langle J_y\rangle_\psi}_{=0} + \langle J_z\rangle\cos\beta$$

$$\langle \vec{J}\cdot\hat{n}\rangle_\psi = \langle\psi|J_z|\psi\rangle\,.\cos\beta$$

$$= \hbar m\,\langle\psi|\psi\rangle\,.\cos\beta$$

$$\boxed{\langle \vec{J}\cdot\hat{n}\rangle_\psi = \hbar\,.\,m\,.\cos\beta}$$

Donde θ del enunciado debe ser ángulo entre $\hat{z}$ y $\hat{n}$, no entre $\hat{x}$ y $\hat{n}$

14.

base $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ $\qquad j=1 \quad m=-1,0,1$

$$L^2|j,m\rangle = \hbar^2(j(j+1))|j,m\rangle$$
$$L_z|j,m\rangle = m\hbar|j,m\rangle$$

$$L^2 = \begin{array}{c c} \begin{array}{ccc} -1 & 0 & 1 \end{array} & \\ \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix} & \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \end{array}$$

$$L_z = \begin{array}{c} \begin{array}{ccc} -1 & 0 & \end{array} \\ \begin{pmatrix} -\hbar\langle 1,-1|1,-1\rangle & 0.\overbrace{\langle 1,-1|1,0\rangle}^{=0} & \hbar\overbrace{\langle 1,-1|1,1\rangle}^{=0} \\ 0 & 0\hbar\langle 1,0|1,0\rangle & 0 \\ 0 & 0 & \hbar\langle 1,1|1,1\rangle \end{pmatrix} \end{array} = \begin{pmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +\hbar \end{pmatrix}$$

L^2, L_z son fáciles porque están en su base de autoestados

$$\boxed{L^2 = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar\,\mathbb{1}} \quad ; \quad \boxed{L_z = \hbar\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$L_x = \frac{(L_+) + (L_-)}{2}$$

$$L_y = \frac{L_+ - L_-}{2i} \longrightarrow$$

$$L_\pm|1,\pm1\rangle = 0$$

las esquinas son nulas

$$L_y = \begin{array}{cc} \begin{array}{ccc} m=-1 & m=0 & m=1 \end{array} & \\ \begin{pmatrix} 0 & +\frac{\sqrt{2}\hbar}{2i} & 0 \\ -\frac{\sqrt{2}\hbar}{2i} & \frac{\sqrt{2}\hbar - \sqrt{2}\hbar}{2i} & \frac{\sqrt{2}\hbar}{2i} \\ 0 & -\frac{\sqrt{2}\hbar}{2i} & 0 \end{pmatrix} & \begin{array}{c} m'=-1 \\ m'=0 \\ m'=1 \end{array} \end{array}$$

$$\boxed{L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}}$$

$$\langle 1,1|\frac{L_+ - L_-}{2i}|1,0\rangle = \frac{\sqrt{2}\hbar}{2i}$$

$$\langle 1,0|\frac{L_+ - L_-}{2i}|1,-1\rangle = \frac{\langle 1,0|}{2i}\left(\hbar\sqrt{2}\,|1,0\rangle + \hbar\sqrt{0}\,|0\rangle\right)$$

$$\langle 1,0|\frac{L_+ - L_-}{2i}|1,1\rangle = \frac{1}{2i}\langle 1,0|\left(\sqrt{0}\hbar\,|1,2\rangle - \sqrt{2}\hbar\,|1,0\rangle\right)$$

$$\langle 1,-1|\frac{L_+ - L_-}{2i}|1,0\rangle = \frac{1}{2i}\langle 1,-1|\left(\sqrt{2}\hbar\,|1,1\rangle - \sqrt{2}\hbar\,|1,-1\rangle\right)$$

$$L_x = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} 0 & \frac{\hbar\sqrt{2}}{2} & 0 \\ \frac{\hbar\sqrt{2}}{2} & 0 & \frac{\hbar\sqrt{2}}{2} \\ 0 & \frac{\hbar\sqrt{2}}{2} & 0 \end{pmatrix} \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \end{array}$$

$$\boxed{L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}}$$

$L_\pm|1,\pm1\rangle = 0$
Las esquinas son nulas

$\frac{1}{2}\langle 1,0|(L_+)+(L_-)|1,-1\rangle = \frac{1}{2}\langle 1,0|(\hbar\sqrt{2}|1,0\rangle + 0)$

$\frac{1}{2}\langle 1,0|(L_+)+(L_-)|1,0\rangle = \frac{1}{2}\langle 1,0|(\quad |1,1\rangle + |1,-1\rangle)$

$\frac{1}{2}\langle 1,0|(L_+)+(L_-)|1,1\rangle = \frac{1}{2}\langle 1,0|(\quad 0 + \frac{\hbar}{\sqrt{2}}|1,0\rangle)$

$\frac{1}{2}\langle 1,-1|(L_+ + L_-)(|1,0\rangle) = \frac{1}{2}\langle 1,-1|(|1,1\rangle + \hbar\sqrt{2}|1,-1\rangle)$

$[L_x, L_y] = L_x L_y - L_y L_x$

$$= \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} - \frac{\hbar^2}{2^2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & -2i \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} -2i & 0 & -2i \\ 0 & 0 & 0 \\ +2i & 0 & +2i \end{pmatrix}$$

$$[L_x, L_y] = \frac{\hbar^2}{4}\begin{pmatrix} 4i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4i \end{pmatrix} = \hbar(\hbar i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar L_z$$

está OK

a) $\{|l, m_j\rangle\}$ base de L_z

$$L_y = \frac{\sqrt{2}\, i\hbar}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hay que diagonalizar la matriz de L_y

$$\frac{i\hbar}{\sqrt{2}}\begin{pmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \rightarrow ((-\lambda)^3 + 0 + 0) + (0 + \lambda + \lambda) =$$

autovalores $-\lambda^3 + 2\lambda = 0$

$(-\lambda^2 - 2)\lambda = 0$

$\lambda^2 = -2$

$\lambda = +\sqrt{2}\, i$, $-\sqrt{2}\, i$

$\lambda = 0$
$\lambda = +\sqrt{2}\, i$
$\lambda = -\sqrt{2}\, i$

__autokets de L_y__

* $\underline{\lambda} = 0$ $\quad V_1 = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \alpha = \frac{1}{\sqrt{2}}$

$\underline{\lambda} = \sqrt{2}i$

$(-\sqrt{2}i)x - y = 0$
$y = -\sqrt{2}\, i x$
$y - \sqrt{2}i z = 0$
$y = \sqrt{2}\, i z$
$x = -z$

$$V_2 = \begin{pmatrix} -z \\ \sqrt{2}i z \\ z \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ \sqrt{2}i \\ 1 \end{pmatrix}$$

$\alpha^2 + \alpha^2 2 + \alpha^2 = 1$
$4\alpha^2 = 1$
$\alpha^2 = \frac{1}{4} \rightarrow \alpha = \frac{1}{2}$

$\underline{\lambda} = -\sqrt{2}i$

$\sqrt{2}ix - y = 0$
$y = \sqrt{2}i\, x$
$y + \sqrt{2}i z = 0$
$y = -\sqrt{2}i\, z$
$x = -z$

$$V_3 = \begin{pmatrix} -z \\ -\sqrt{2}i z \\ z \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{pmatrix} \rightarrow \alpha = \frac{1}{2}$$

AL definir la matriz de L^2 según el arreglo:

$$\left(\begin{array}{ccc|c} \langle 1,-1| & \langle 1,0| & \langle 1,1| & \\ \hline & & & |1,-1\rangle \\ & & & |1,0\rangle \\ & & & |1,1\rangle \\ \hline \end{array}\right) \Rightarrow$$

hemos establecido

$$|1,-1\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{asociado con } -\hbar$$

$$|1,0\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{asociado con } 0$$

$$|1,1\rangle \doteq \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{asociado con } \hbar$$

Base hallada del subespacio

$$B = \left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}}_{V_1} ; \underbrace{\frac{1}{2}\begin{pmatrix}-1\\\sqrt{2}i\\1\end{pmatrix}}_{V_2} ; \underbrace{\frac{1}{2}\begin{pmatrix}-1\\-\sqrt{2}i\\1\end{pmatrix}}_{V_3} \right\}$$

$$V_1 = \frac{1}{\sqrt{2}}\left(|1,-1\rangle + |1,1\rangle \right)$$

$$V_2 = \frac{1}{2}\left(-|1,-1\rangle + \sqrt{2}i\, |1,0\rangle + |1,1\rangle \right)$$

$$V_3 = \frac{1}{2}\left(-|1,-1\rangle - \sqrt{2}i\, |1,0\rangle + |1,1\rangle \right)$$

Acá confirmamos que los autovalores sean los que sospechábamos según las asociaciones

$$L_y|V_1\rangle = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} = 0$$

$$L_y|V_2\rangle = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}-1/2\\\sqrt{2}i/2\\1/2\end{pmatrix}$$

autovalor $-\hbar$ $\quad \frac{1}{\sqrt{2}i}\frac{i\hbar}{\sqrt{2}} \begin{pmatrix}-\frac{\sqrt{2}i}{2}\\-1\\\frac{\sqrt{2}i}{2}\end{pmatrix}$

$$L_y|V_3\rangle = +\hbar\, |V_3\rangle \qquad -\frac{1}{\sqrt{2}i} = \frac{\sqrt{2}i}{2}$$

b) $$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle \right)$$

Medimos L_y $\Rightarrow$ tener que proyectar $|\Psi\rangle$ en la base de autoestados $\{V_1, V_2, V_3\}$ $\Rightarrow$

$$P_{|V_1;0\rangle} = |\langle V_1|\Psi\rangle|^2 = \left| \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\left(\langle 1,-1| + \langle 1,1| \right)\left(|1,1\rangle - |1,-1\rangle \right) \right|^2$$

$$= \left| \frac{1}{2}(1-1) \right|^2 = \boxed{0} \quad \text{prob. de } 0$$

$$P_{|V_2;-\hbar\rangle} = |\langle V_2|\Psi\rangle|^2 = \left| \frac{1}{2}\frac{1}{\sqrt{2}}\left(-\langle 1,-1| - \langle 1,0|\sqrt{2}i + \langle 1,1| \right)\left(|1,1\rangle - |1,-1\rangle \right) \right|^2$$

$$= \left| \frac{1}{\cancel{2}}\frac{1}{\sqrt{2}}\cancel{2} \right|^2 = \boxed{\frac{1}{2}} \quad \text{prob. de } -\hbar$$

$$P_{|V_3;+\hbar\rangle} = |\langle V_3|\Psi\rangle|^2 = \left| \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\left(-\langle 1,-1| + \langle 1,0|\sqrt{2}i + \langle 1,1| \right)\left(|1,1\rangle - |1,-1\rangle \right) \right|^2$$

$$= \left| \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\cdot(1+1) \right|^2 = \boxed{\frac{1}{2}} \quad \text{prob. de } +\hbar$$

Midamos L_x

Ahora habría que diagonalizar L_x, para obtener su base de autoestados

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0$$

$$\left((-\lambda)^3+0+0\right)-\left(0-\lambda-\lambda\right)=0$$
$$-\lambda^3+2\lambda=0$$
$$\lambda^2-2=0$$

$\lambda=0$
$\lambda=\sqrt{2}$
$\lambda=-\sqrt{2}$

* $\lambda=0$; V_1

$y=0$

$x+z=0$

$$V_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

* $\lambda=\sqrt{2}$; V_2

$-\sqrt{2}\,x+y=0$

$y=\sqrt{2}\,x$

$y-\sqrt{2}\,z=0$

$y=\sqrt{2}\,z$

$$V_2=\alpha\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} \qquad \alpha^2(1+2+1)=1 \qquad \alpha=\frac{1}{\sqrt{4}}=\frac{1}{2}$$

$$V_2=\frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

* $\lambda=-\sqrt{2}$; V_3

$\sqrt{2}x+y=0$

$y=-\sqrt{2}x$

$y+\sqrt{2}z=0$

$y=-\sqrt{2}z$

$$V_3=\alpha\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

$$V_1=\frac{1}{\sqrt{2}}\left(|1,-1\rangle-|1,1\rangle\right) \ ; \quad V_2=\frac{1}{2}\left(|1,-1\rangle+\sqrt{2}\,|1,0\rangle+|1,1\rangle\right)$$

$$V_3=\frac{1}{2}\left(|1,-1\rangle-\sqrt{2}\,|1,0\rangle+|1,1\rangle\right)$$

$$L_x|V_2\rangle=\frac{\hbar}{2}\sqrt{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1/2\\ \sqrt{2}/2\\ 1/2\end{pmatrix}=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}\sqrt{2}/2\\ 1\\ \sqrt{2}/2\end{pmatrix}=+\hbar\,|V_2\rangle \qquad (+\hbar)\begin{pmatrix}1/2\\ 1/\sqrt{2}\\ 1/2\end{pmatrix}$$

$$P_{|V_2,+\hbar\rangle}=\left|\frac{1}{2}\left(\langle 1,-1|+\sqrt{2}\langle 1,0|+\langle 1,1|\right)\left(|1,1\rangle-|1,-1\rangle\right)\frac{1}{\sqrt{2}}\right|^2$$
$$=\left|\frac{1}{2}\,\frac{1}{2^{1/2}}\left(1+(-1)\right)\right|^2=\boxed{0}$$ no podemos medir $+\hbar$

$$P_{|V_3,-\hbar\rangle}=\left|\frac{1}{2}\left(\langle 1,-1|-\sqrt{2}\langle 1,0|+\langle 1,1|\right)\left(|1,1\rangle-|1,-1\rangle\right)\frac{1}{\sqrt{2}}\right|^2=\left|\frac{1}{2}\,\frac{1}{\sqrt{2}}\,(-1+1)\right|^2=\boxed{0}$$ no medimos $-\hbar$

$$P_{(V_1,0)}=\left|\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\left(-\langle 1,1|+\langle 1,-1|\right)\left(|1,1\rangle-|1,-1\rangle\right)\right|^2=\left|\frac{1}{2}(-1-1)\right|^2=\boxed{1}$$ Medimos cero

c) Dada $|\psi\rangle=\frac{1}{\sqrt{2}}|1,1\rangle-\frac{1}{\sqrt{2}}|1,-1\rangle$ medimos $\hbar$ ⇒ mirando la matriz el AV. corresp es el $|1,1\rangle$ ⇒

estamos en $|\psi\rangle_{\text{Después de medir}}=|1,1\rangle$

Midamos ahora L_y

$$P_{(V_1;0)}=\left|\frac{1}{\sqrt{2}}\left(\langle 1,1|+\langle 1,-1|\right)\left(|1,1\rangle\right)\right|^2=\left|\frac{1}{\sqrt{2}}\right|^2=\boxed{\frac{1}{2}}$$

$$P(V_2,-\hbar)=\left|\frac{1}{2}\left(-\langle 1,-1|-\sqrt{2}\,i\langle 1,0|+\langle 1,1|\right)\left(|1,1\rangle\right)\right|^2=\left|\frac{1}{2}\right|^2=\boxed{\frac{1}{4}}$$

$$P(V_1;+\hbar)=\left|\frac{1}{2}\left(-\langle 1,-1|+\sqrt{2}\,i\langle 1,0|+\langle 1,1|\right)\left(|1,1\rangle\right)\right|^2=\left|\frac{1}{2}\right|^2=\boxed{\frac{1}{4}}$$

Podremos obtener 0 con prob. $1/2$
$-\hbar$ con prob. $1/4$
$+\hbar$ con prob. $1/4$

15. $|j,j\rangle$ hasta términos de orden ε^2

$$\mathcal{D}_{\hat{y}\varepsilon}|j,j\rangle = e^{-\frac{i\vec{J}\cdot\hat{y}\varepsilon}{\hbar}}|j,j\rangle = e^{-\frac{iJ_y\varepsilon}{\hbar}}|j,j\rangle$$

$$\mathcal{D}_{\hat{y}\varepsilon} \approx 1 + \frac{-iJ_y\varepsilon}{\hbar} + \left(\frac{-iJ_y\varepsilon}{\hbar}\right)^2\frac{1}{2} \Rightarrow$$

$$|j,j\rangle_R \approx \left(1 - \frac{iJ_y\varepsilon}{\hbar} + \frac{J_y^2\varepsilon^2}{2\hbar^2}\right)|j,j\rangle$$

$$|j,j\rangle_R \approx |j,j\rangle - \frac{i\varepsilon}{\hbar}J_y|j,j\rangle + \frac{\varepsilon^2}{2\hbar^2}J_y^2|j,j\rangle$$

$$|j,j\rangle_R \approx |j,j\rangle - \frac{i\varepsilon}{\hbar}\left(\frac{J_+ - J_-}{2i}\right)|j,j\rangle + \frac{\varepsilon^2}{2\hbar^2}\left(\frac{(J_+ - J_-)(J_+ - J_-)}{4i^2}\right)|j,j\rangle$$

$$\approx |j,j\rangle - \frac{\varepsilon}{2\hbar}(J_+ - J_-)|j,j\rangle - \frac{\varepsilon^2}{8\hbar^2}(J_+J_+ - J_-J_+ - J_+J_- + J_-J_-)|j,j\rangle$$

$$\approx |j,j\rangle - \frac{\varepsilon}{2\hbar}\Big(-\hbar\sqrt{\underbrace{j(j+1)-j(j-1)}_{j^2+j-j^2+j}}\Big)|j,j-1\rangle - \frac{\varepsilon^2}{8\hbar^2}\Big([J_+ - J_-]\underbrace{J_+|j,j\rangle}_{0}$$

$$+\underbrace{(-J_+ + J_-)J_-|j,j\rangle}\Big)$$

$$(-J_+ + J_-)\left(\hbar\sqrt{j(j+1)-j(j-1)}\right)|j,j-1\rangle =$$

$$= +\sqrt{j(j+1)-j(j-1)}\,\hbar^2\sqrt{j(j+1)-(j-1)[(j-1)-1]}\,|j,j-2\rangle$$

$$-\sqrt{j(j+1)-j(j-1)}\,\hbar^2\sqrt{j(j+1)-(j-1)[(j-1)+1]}\,|j,j\rangle$$

$4j-2$

$j(j+1)-(j-1)(j-2)$

$j^2+j-(j^2-j-2j+2)$

$j(j+1)-(j-1)(j)$

j^2+j-j^2+j

$$\approx |j,j\rangle + \frac{\varepsilon\hbar}{2\hbar}\sqrt{j2}\,|j,j-1\rangle - \frac{\varepsilon^2}{8\cancel{\hbar^2}}\cancel{\hbar^2}\left(\sqrt{j2}\sqrt{2(j-1)}\,|j,j-2\rangle + \sqrt{j2}\sqrt{2j}\,|j,j\rangle\right)$$

$$\approx |j,j\rangle + \frac{\varepsilon\sqrt{j}}{\sqrt{2}}|j,j-1\rangle - \frac{\varepsilon^2\sqrt{2(j-1)}}{4}|j,j-2\rangle - \frac{\varepsilon^2}{4}j\,|j,j\rangle$$

$$\underset{\text{orden }\varepsilon^2}{|j,j\rangle_R} = \left(1 - \frac{\varepsilon^2}{4}j\right)|j,j\rangle + \frac{\varepsilon\sqrt{j}}{\sqrt{2}}|j,j-1\rangle - \frac{\varepsilon^2}{4}\sqrt{2(j-1)}\,|j,j-2\rangle$$

$$|\langle j,j|j,j\rangle_R|^2 = |\langle j,j|\left(1-\frac{\varepsilon^2}{4}j\right)|j,j\rangle|^2 = \left|1 - \frac{\varepsilon^2}{4}j\right|^2 = 1 - \frac{2\varepsilon^2 j}{4} + \frac{\varepsilon^4 j^2}{16}$$

solo sobreviven los términos en $|j,j\rangle$ $\longrightarrow$

$$\boxed{P_{|j,j\rangle} \approx 1 - \frac{\varepsilon^2}{2}j + \mathcal{O}(\varepsilon^4)}$$

16.

$$(G_i)_{jk} = -i\hbar\, \varepsilon_{ijk} \qquad \varepsilon_{1\,\substack{23\\32}} \qquad \varepsilon_{2\,\substack{13\\31}} \qquad \varepsilon_{3\,\substack{12\\21}}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix} \qquad G_2 = \begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix} \qquad G_3 = \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[G_i, G_j] = i\hbar\, G_k\, \varepsilon_{ijk}$$

$$G_1 G_2 - G_2 G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ (-i\hbar)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (i\hbar)^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\hbar^2 & 0 \\ -\hbar^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[G_1, G_2] = i\hbar\, G_3 = i\hbar \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hbar^2 & 0 \\ -\hbar^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ satisface

$$[G_\ell, G_m]_{jk} = (-i\hbar\, \varepsilon_{\ell jk})(-i\hbar\, \varepsilon_{mkj}) - (-i\hbar\, \varepsilon_{mjk})(-i\hbar\, \varepsilon_{\ell kj})$$

$$= (-i\hbar)\,[\varepsilon_{\ell jk}\varepsilon_{mkj} - \varepsilon_{mjk}\varepsilon_{\ell kj}]$$

$$[\varepsilon_{jk\ell}\,\varepsilon_{jmk} - \varepsilon_{jkm}\,\varepsilon_{j\ell k}]$$

$$[\delta_{km}\delta_{\ell k} - \delta_{kk}\delta_{\ell m} - (\delta_{k\ell}\delta_{mk} - \delta_{kk}\delta_{m\ell})]$$

$$[\delta_{km}\delta_{\ell k} - \delta_{k\ell}\delta_{mk}] \qquad k=\ell=m$$

17. $V(r) \rightarrow$ H esféricamente simétrico

$$\Psi(x) = (x+y+3z)\, f(r)$$

$$\Psi(r,\theta,\phi) = r.\left(\cos\phi . \operatorname{sen}\theta + \operatorname{sen}\phi . \operatorname{sen}\theta + 3.\cos\theta\right) f(r)$$

$$\Psi(r,\theta,\phi) = \underbrace{r.f(r)}_{=R(r)} . \left(\cos\phi . \operatorname{sen}\theta + \operatorname{sen}\phi . \operatorname{sen}\theta + 3.\cos\theta\right) = \langle r,\theta,\phi | n,l,m\rangle$$

$$\sum_l \sum_m \langle r,\theta,\phi|\hat{n}\rangle \underbrace{\langle \hat{n}|n,l,m\rangle}_{Y_l^m(\hat{n})}$$

$$\Psi(r,\theta,\phi) = R(r) . \sum_l \sum_{m=-l}^{l} Y_l^m(\theta,\phi)$$

queremos ver hasta donde va l

a)

$$\sqrt{\frac{3}{8\pi}} . \operatorname{sen}\theta . (\cos\phi + i.\operatorname{sen}\phi) = Y_1^1$$

$$\sqrt{\frac{3}{8\pi}} \operatorname{sen}\theta (\cos\phi - i.\operatorname{sen}\phi) = Y_1^{-1}$$

$$\rightarrow \quad Y_1^{-1} - Y_1^1 = 2\sqrt{\frac{3}{8\pi}} \operatorname{sen}\theta . \cos\phi$$

$$Y_1^1 + Y_1^{-1} = -2i . \sqrt{\frac{3}{8\pi}} \operatorname{sen}\theta . \operatorname{sen}\phi$$

$$\sqrt{\frac{3}{4\pi}} . \cos\theta = Y_1^0$$

$$3.\cos\theta = \sqrt{3.4\pi}\; Y_1^0 = \sqrt{12\pi}\; Y_1^0$$

$$\Psi(r,\theta,\phi) = R(r) . \left[\sqrt{\frac{8\pi}{3}} \frac{(Y_1^{-1} - Y_1^1)}{2} - \sqrt{\frac{8\pi}{3}} \frac{1}{2i} (Y_1^1 + Y_1^{-1}) + \sqrt{12\pi}\; Y_1^0 \right]$$

$$\left[\sqrt{\frac{2\pi}{3}} Y_1^{-1} - \sqrt{\frac{2\pi}{3}} Y_1^1 + i\sqrt{\frac{2\pi}{3}} Y_1^1 + i\sqrt{\frac{2\pi}{3}} Y_1^{-1} + \sqrt{12\pi}\; Y_1^0 \right]$$

$$\left[\sqrt{\frac{2\pi}{3}} (1+i)\; Y_1^{-1} + \sqrt{\frac{2\pi}{3}} (-1+i)\; Y_1^1 + \sqrt{12\pi}\; Y_1^0 \right]$$

$$L^2 \left(\qquad \right) + L^2 \left(\qquad \right) + L^2 \left(\sqrt{12\pi}\; Y_1^0 \right)$$

$$\left[2\hbar^2 \left(\qquad \right) Y_1^{-1} + 2\hbar^2 \left(\qquad \right) Y_1^1 + 2\hbar^2 \left(\qquad \right) Y_1^0 \right]$$

$\Rightarrow$ es autofunción de L^2 con autovalor $\boxed{2\hbar^2}$

con $\boxed{l = 1}$

b)

$$m = 0 \qquad P_{m=0} = |\Psi_{m=0}(r,\theta,\phi)|^2$$

$$P_{m=1} = |\Psi_{m=1}(r,\theta,\phi)|^2$$

$$P_{m=-1} = |\Psi_{m=-1}(r,\theta,\phi)|^2$$

Como la parte radial es la misma sólo necesitamos normalizar con los $Y_l^m(\theta,\phi)$ $\rightarrow$

$$1 = |\langle Y_1^0 | Y_l^m \rangle|^2 + |\langle Y_1^{-1} | Y_l^m \rangle|^2 + |\langle Y_1^1 | Y_l^m \rangle|^2$$

$$\left|\sqrt{\frac{2\pi}{3}}(1+i)\right|^2 + \left|\sqrt{\frac{2\pi}{3}}(-1+i)\right|^2 + \left|\sqrt{12\pi}\right|^2 = 1$$

$$\alpha^2\left(\left|\sqrt{\frac{2\pi}{3}}\right|\left|\sqrt{2}\right|^2 + \left|\sqrt{\frac{2\pi}{3}}\right|^2\left|\sqrt{2}\right|^2 + 12\pi\right) = 1$$

$$\alpha^2\left(\frac{2\pi}{3}2 + \frac{2\pi}{3}.2 + 12\pi\right) = 1$$

$$\alpha^2\left(\frac{44}{3}.\pi\right) = 1$$

$$\alpha = \sqrt{\frac{3}{44.\pi}} \rightarrow$$

$$P_{(m=0)} = \cancel{12}^{3}\cancel{\pi}.\frac{3}{\cancel{44\pi}_{11}} = \boxed{\frac{9}{11}}$$

$$P_{(m=1)} = \frac{\cancel{4}^{1}\pi}{\cancel{3}}\frac{\cancel{3}}{\cancel{44\pi}_{11}} = \boxed{\frac{1}{11}} \qquad P_{(m=-1)} = \boxed{\frac{1}{11}}$$

c)

$$H\,\Psi(x) = E\,\Psi(x)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(x) - E\,\Psi(x) = 0$$

$$-\frac{\hbar^2}{2m}\nabla_r^2\,\Psi(x) - \frac{\hbar^2}{2m}\nabla_{\theta,\phi}^2\,\Psi(x) + V(r)\Psi(x) - E\,\Psi(x) = 0$$

$$-\frac{\hbar^2}{2m}\nabla_r^2 R(r).\cancel{f(\theta,\phi)} - \frac{\hbar^2}{2m}\nabla_{\phi,\theta}^2 R(r).f(\theta,\phi) + V(r).R(r).\cancel{f(\theta,\phi)} - E\,R(r)\cancel{f(\theta,\phi)} = 0$$

$$+\frac{L^2}{2mr^2}R(r).f(\theta,\phi)$$

$$+\frac{\ell(\ell+1)\hbar^2}{2m\,r^2}R(r).\cancel{f(\theta,\phi)}$$

$$\boxed{-\frac{\hbar^2}{2m}\nabla_r^2 R(r) + \left[V(r) - E + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]R(r) = 0}$$

Resolviendo esta ecuación puede hallarse V(r)

19.

a) $j=1$

$$\langle 1,m' | J_y | 1,m\rangle$$
$$\downarrow$$
$$\langle 1,m' | \frac{J_+ - J_-}{2i} | 1,m\rangle$$

$$\langle 1,0 | \frac{L_+ - L_-}{2i} | 1,0\rangle = \langle 1,0| \left(\# |1,1\rangle - \# |1,-1\rangle \right) = 0$$
$$\langle 1,0 | \frac{L_+ - L_-}{2i} | 1,1\rangle = \frac{\langle 1,0|}{2i} \left(\# |1,2\rangle - \hbar\sqrt{2} |1,0\rangle \right) = -\frac{\hbar\sqrt{2}}{2i}$$
$$\langle 1,0 | \frac{L_+ - L_-}{2i} | 1,-1\rangle = \frac{\langle 1,0|}{2i} \left(\hbar\sqrt{2} |1,0\rangle - \# 0 \right) = \frac{\hbar\sqrt{2}}{2i}$$
$$\langle 1,1 | \frac{L_+ - L_-}{2i} | 1,0\rangle = \frac{\langle 1,1|}{2i} \left(\hbar\sqrt{2} |1,1\rangle - \# |1,-1\rangle \right) = \frac{\hbar\sqrt{2}}{2i}$$
$$\langle 1,1 | \frac{L_+ - L_-}{2i} | 1,1\rangle = \frac{\langle 1,1|}{2i} \left(\#.0 - \# |1,0\rangle \right) = 0$$
$$\langle 1,1 | \frac{L_+ - L_-}{2i} | 1,-1\rangle = \frac{\langle 1,1|}{2i} \left(\# |1,0\rangle - \# 0 \right) = 0$$
$$\langle 1,-1 | \frac{L_+ - L_-}{2i} | 1,0\rangle = \frac{\langle 1,-1|}{2i} \left(\# |1,1\rangle - \hbar\sqrt{2} |1,-1\rangle \right) = \frac{\hbar\sqrt{2}}{2i}$$
$$\langle 1,-1 | \frac{L_+ - L_-}{2i} | 1,1\rangle = \frac{\langle 1,-1|}{2i} \left(\#.0 - \# |1,0\rangle \right) = 0$$
$$\langle 1,-1 | \frac{L_+ - L_-}{2i} | 1,-1\rangle = \frac{\langle 1,-1|}{2i} \left(\# |1,0\rangle - \#.0 \right) = 0$$

$$J_- |1,1\rangle = \hbar\sqrt{2} |1,0\rangle$$
$$J_+ |1,-1\rangle = \hbar\sqrt{2} |1,0\rangle$$
$$J_+ |1,0\rangle = \hbar\sqrt{2} |1,1\rangle$$
$$J_- |1,0\rangle = \hbar\sqrt{2} |1,-1\rangle$$

$$J_y \doteq \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} 0 & \frac{\hbar\sqrt{2}}{2i} & 0 \\ \frac{\hbar\sqrt{2}}{2i} & 0 & -\frac{\hbar\sqrt{2}}{2i} \\ 0 & -\frac{\hbar\sqrt{2}}{2i} & 0 \end{pmatrix} \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \end{array}$$

$$\boxed{J_y \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ -\sqrt{2}i & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}}$$

b) caso $j=1$

$$e^{-iJ_y\beta/\hbar} = \sum_{n=0}^{\infty} \left(\frac{i J_y \beta}{\hbar} \right)^n \frac{1}{n!}$$

$i^2 i = -i$

$$e^{-iJ_y\beta/\hbar} \approx \underbrace{1}_{} + \underbrace{\frac{-i J_y \beta}{\hbar}}_{} + \underbrace{\frac{1}{2} \frac{(-i)^2 J_y^2 \beta^2}{\hbar^2}}_{} + \underbrace{\frac{1}{3!} \frac{(-i)^3 J_y^3 \beta^3}{\hbar^3}}_{} + \frac{1}{4!} \frac{(-i)^4 J_y^4 \beta^4}{\hbar^4}$$

$(-i)^{2n} = [(-i)^2]^n$

$\neq 1$

$(-i)(-i)$

i^2

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{(-i)^{2n} J_y^{2n} \beta^{2n}}{\hbar^{2n}} + \sum_{n=0}^{\infty} \frac{(-i)^{2n+1} J_y^{2n+1} \beta^{2n+1}}{\hbar^{2n+1}} \cdot \frac{1}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n J_y^{2n}}{(2n)!} \left(\frac{\beta}{\hbar} \right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n (-i)}{(2n+1)!} \left(\frac{J_y \beta}{\hbar} \right)^{2n+1}$$

$$J_y = \left(\frac{\hbar i}{\sqrt{2}}\right)\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y^2 = \left(\frac{\hbar}{2}\sqrt{2}\, i\right)^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\left(\frac{\hbar i}{\sqrt{2}}\right)^2 = \left(-\frac{\hbar^2}{2}\right)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$J_y^3 = \left(\quad\right)^3 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\left(\frac{\hbar i}{\sqrt{2}}\right)^3 = 2\,(J_y)\left(\frac{\hbar i}{\sqrt{2}}\right)^2 = -\cancel{2} J_y \frac{\hbar^2}{\cancel{2}}$$

$$J_y^4 = \left(\quad\right)^4 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}\left(\frac{\hbar i}{\sqrt{2}}\right)^4 = 2 J_y^2 \left(\frac{\hbar i}{\sqrt{2}}\right)^2 = -J_y^2 \hbar^2$$

$$J_y^6 = \left(\quad\right)^6 \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 8 & 0 \\ -4 & 0 & 4 \end{pmatrix}\left(\frac{\hbar i}{\sqrt{2}}\right)^6 = 4 J_y^2 \left(\frac{\hbar i}{\sqrt{2}}\right)^4 = J_y^2 \hbar^4$$

$$J_y^5 = \left(\frac{\hbar i}{\sqrt{2}}\right)^5 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} 2 = J_y^3 \cdot 2 \left(\frac{\hbar i}{\sqrt{2}}\right)^2 = 2 J_y\, 2 \left(\frac{\hbar i}{\sqrt{2}}\right)^4 = \cancel{4} J_y \hbar^4 \cancel{4}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n J_y^{2n}}{(2n)!}\left(\frac{\beta}{\hbar}\right)^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} J_y^{2n+1} \left(\frac{\beta}{\hbar}\right)^{2n+1}$$

$n \geq 1$

$$J_y^{2n} = 2^{n-1} J_y^2 \left(\frac{\hbar i}{\sqrt{2}}\right)^{2n-2} \qquad J_y^{2n+1} = 2^n \left(\frac{\hbar i}{2^{1/2}}\right)^{2n} J_y \quad n \geq 1$$

$$\left(\frac{J_y}{\hbar}\right)^{2n} = \cancel{2^{n-1}} \frac{J_y^2}{\hbar^2} \frac{(i^2)^{n-1}}{\cancel{2^{n-1}}} \qquad \left(\frac{J_y}{\hbar}\right)^{2n+1} = \cancel{2^n} \frac{i^{2n}}{\cancel{2^n}} \frac{J_y}{\hbar}$$

$$\left(\frac{J_y}{\hbar}\right)^{2n} = \frac{J_y^2}{\hbar^2}(-1)^{n-1} \qquad \left(\frac{J_y}{\hbar}\right)^{2n+1} = (-1)^n \left(\frac{J_y}{\hbar}\right)$$

$$J_y^7 = \left(\quad\right)^7 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} 4 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \left(\frac{\hbar i}{\sqrt{2}}\right)^4 4\, J_y^3 = 2\cdot 4\, J_y \left(\frac{\hbar i}{\sqrt{2}}\right)^6 = 8\, J_y \left(\frac{\hbar i}{\sqrt{2}}\right)^6 = -J_y \hbar^6$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!}\left(\frac{J_y \beta}{\hbar}\right)^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!}\left(\frac{J_y \beta}{\hbar}\right)^{2n+1}$$

$$1 + \sum_{n=1}^{\infty} \frac{i^{2n}}{(2n)!}\left(\frac{J_y \beta}{\hbar}\right)^{2n}$$

$$(i^2)^n \quad (-1)^n = (-1)^n (-1)^n \quad (-1)^{2n}$$

$$1 + \left(\frac{J_y^2}{\hbar^2}\right)\Bigg[\underbrace{\sum_{n=1}^{\infty} -\frac{i^{2n}(-1)^n}{(2n)!}\beta^{2n}}_{-\cos i\beta}\Bigg] + i\left(\frac{J_y}{\hbar}\right)\sum_{n=0}^{\infty} \frac{i^{2n}(-1)^n}{(2n+1)!}\beta^{2n+1}$$

c)

$$j=1 \qquad e^{-\frac{iJ_y\beta}{\hbar}} = 1 - i\left(\frac{J_y}{\hbar}\right)\operatorname{sen}\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

$$d^{j=1}(\beta) = \begin{pmatrix} \langle 1,-1| e^{-\frac{iJ_y\beta}{\hbar}} |1,-1\rangle & \langle 1,-1| e^{-\frac{iJ_y\beta}{\hbar}} |1,0\rangle & \langle 1,-1| e^{-\frac{iJ_y\beta}{\hbar}} |1,1\rangle \\ \langle 1,0| e^{-\frac{iJ_y\beta}{\hbar}} |1,-1\rangle & \langle 1,0| e^{-\frac{iJ_y\beta}{\hbar}} |1,0\rangle & \langle 1,0| e^{-\frac{iJ_y\beta}{\hbar}} |1,1\rangle \\ \langle 1,1| e^{-\frac{iJ_y\beta}{\hbar}} |1,-1\rangle & \langle 1,1| e^{-\frac{iJ_y\beta}{\hbar}} |1,0\rangle & \langle 1,1| e^{-\frac{iJ_y\beta}{\hbar}} |1,1\rangle \end{pmatrix}$$

$$\frac{J_y^2}{\hbar^2} = \begin{array}{c} \begin{array}{ccc} -1\rangle & 0\rangle & 1\rangle \end{array} \\ \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \end{array} \begin{array}{l} \langle -1 \\ \langle 0 \\ \langle 1 \end{array} \qquad \frac{J_y}{\hbar} = \begin{array}{c} \begin{array}{ccc} -1\rangle & 0\rangle & 1\rangle \end{array} \\ \begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ -i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix} \end{array} \langle 0$$

$$d^{j=1}_{11}(\beta) = 1 - \frac{1}{2}(1-\cos\beta) = 1 - \frac{1}{2} + \frac{\cos\beta}{2} \qquad d^{j=1}_{12}(\beta) = -i\left(\frac{-i}{\sqrt{2}}\right)\operatorname{sen}\beta = -\frac{1}{\sqrt{2}}\operatorname{sen}\beta$$

$$d^{j=1}_{13}(\beta) = -\left(-\frac{1}{2}\right)(1-\cos\beta) = \qquad d^{j=1}_{21}(\beta) = -i\left(\frac{-i}{\sqrt{2}}\right)\operatorname{sen}\beta = -\frac{1}{\sqrt{2}}\operatorname{sen}\beta$$

$$d^{j=1}_{22}(\beta) = 1-(1-\cos\beta) = \cos\beta \qquad d^{j=1}_{23}(\beta) = -i\left(\frac{i}{\sqrt{2}}\right)\operatorname{sen}\beta = \frac{1}{\sqrt{2}}\operatorname{sen}\beta$$

$$d^{j=1}_{31}(\beta) = -\left(-\frac{1}{2}\right)(1-\cos\beta) = \qquad d^{j=1}_{32}(\beta) = -i\left(\frac{i}{\sqrt{2}}\right)\operatorname{sen}\beta = \frac{1}{\sqrt{2}}\operatorname{sen}\beta$$

$$d^{j=1}_{33}(\beta) = 1 - \frac{1}{2}(1-\cos\beta) = 1 - \frac{1}{2} + \frac{\cos\beta}{2}$$

$$\boxed{d^{j=1}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\operatorname{sen}\beta & \frac{1}{2}(1-\cos\beta) \\ -\frac{1}{\sqrt{2}}\operatorname{sen}\beta & \cos\beta & \frac{1}{\sqrt{2}}\operatorname{sen}\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\operatorname{sen}\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}}$$