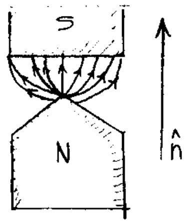
PRÁCTICA 1 : Estados cuánticos, Operadores y Espectros.

1. a) Porque en el sistema físico que corresponde a tener átomos de Ag penetrando en un dispositivo de Stern-Gerlach orientado en $\hat{n}$, a la salida solo pueden tener uno de dos valores posibles



$|S\hat{n};+\rangle$ ó bien $|S\hat{n};-\rangle$
"spin arriba" "spin abajo"

$\Rightarrow$ **N=2 es el total de estados para cualquier átomo**

b) Porque cualquier estado de espín de los átomos de Ag emergiendo de un dispositivo de Stern-Gerlach puede escribirse como CL de $|S_z,+\rangle$ y $|S_z,-\rangle$. (†)

c) Responde a que utilizando coeficientes reales no pueden representarse ciertos estados.

2.

$$S_x|+\rangle = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)|+\rangle = \frac{\hbar}{2}\Big[|+\rangle\underbrace{\langle-|+\rangle}_{=0} + |-\rangle\langle+|+\rangle\Big] = \frac{\hbar}{2}|-\rangle$$

$$S_x|-\rangle = \frac{\hbar}{2}\Big[|+\rangle\langle-|-\rangle + |-\rangle\underbrace{\langle+|-\rangle}_{=0}\Big] = \frac{\hbar}{2}|+\rangle$$

$$S_y|+\rangle = i\frac{\hbar}{2}\Big[-|+\rangle\underbrace{\langle-|+\rangle}_{=0} + |-\rangle\langle+|+\rangle\Big] = i\frac{\hbar}{2}|-\rangle$$

$$S_y|-\rangle = i\frac{\hbar}{2}\Big[-|+\rangle\langle-|-\rangle + |-\rangle\underbrace{\langle+|-\rangle}_{=0}\Big] = -i\frac{\hbar}{2}|+\rangle$$

$$S_z|+\rangle = \frac{\hbar}{2}\Big[|+\rangle\langle+|+\rangle - |-\rangle\underbrace{\langle-|+\rangle}_{=0}\Big] = \frac{\hbar}{2}|+\rangle$$

$$S_z|-\rangle = \frac{\hbar}{2}\Big[|+\rangle\underbrace{\langle+|-\rangle}_{=0} - |-\rangle\langle-|-\rangle\Big] = -\frac{\hbar}{2}|-\rangle$$

Podríamos pasar a matrices los operadores $\Rightarrow$

$$(S_y)_{11} = \frac{i\hbar}{2}\langle+|\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)|+\rangle = \frac{i\hbar}{2}\left(-\langle-|+\rangle\right) = 0$$

$$(S_y)_{12} = \frac{i\hbar}{2}\langle+|\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)|-\rangle = \frac{i\hbar}{2}\left(-\langle-|-\rangle\right) = -\frac{i\hbar}{2}$$

$$S_y \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{i\hbar}{2} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(S_z)_{11} = \frac{\hbar}{2}\langle+|\left(|+\rangle\langle+| - |-\rangle\langle-|\right)|+\rangle = \frac{\hbar}{2}\left(\langle+|+\rangle\right) = 1.\hbar/2$$

$$(S_z)_{12} = \frac{\hbar}{2}\langle+|\left(|+\rangle\langle+| - |-\rangle\langle-|\right)|-\rangle = \frac{\hbar}{2}\langle+|-\rangle = 0$$

$$(S_z)_{22} = \hbar/2\,\langle-|\left(|+\rangle\langle+| - |-\rangle\langle-|\right)|-\rangle = \hbar/2\left(-\langle-|-\rangle\right) = -\hbar/2$$

(†) El modo de explicar los resultados experimentales obtenidos con diversos arreglos de aparatos SG es suponer que el estado de Spin es un ente vectorial y por ende tiene proyecciones en una dada base ortogonal.

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{i\hbar}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$-\frac{1}{i} = i$$

$$S_x S_y = \left(\frac{\hbar}{2}\right)^2 (i) \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$S_x S_z = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\left(\frac{\hbar}{2}\right) S_y$$

$$= \frac{i\hbar}{2} S_y \qquad i\frac{\hbar}{2} i\left(\frac{\hbar}{2}\right)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$S_x S_x = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \left(\frac{\hbar}{2}\right)^2 \mathbb{I} \quad \Rightarrow$$

$$; \quad S_z S_x = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left(\frac{\hbar}{2}\right)^2 = -\frac{\hbar}{2}\left(\frac{\hbar}{2}\right)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{i}{i} =$$

$$[S_i, S_j] = S_i S_j - S_j S_i = 0 \quad \text{si} \quad i=j \quad \Rightarrow \quad [S_i, S_j]_{i\neq j} = \varepsilon_{ijk} \cdot A_k$$

↓ algún comp. de operador

$$S_y S_z = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\frac{\hbar}{2} S_x$$

$$S_z S_y = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = -i\frac{\hbar}{2} S_x$$

$$S_y S_x = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -S_x S_y$$

$$[S_x, S_z] = -i\hbar S_y \qquad [S_z, S_x] = i\hbar S_y$$

$$[S_y, S_z] = i\hbar S_x \qquad [S_z, S_y] = -i\hbar S_x$$

$$[S_x, S_y] = 2 \cdot S_x S_y = i\hbar \cdot \left(\frac{\hbar}{2}\right)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -i\frac{\hbar}{2} S_z$$

$$[S_y, S_x] = i\frac{\hbar}{2} S_z$$

$$[S_x, S_z] = i \cdot \underbrace{\varepsilon_{xzy}}_{=-1} \hbar S_y \quad ; \quad [S_z, S_x] = i \cdot \underbrace{\varepsilon_{zxy}}_{=1} \hbar S_y$$

Juntando todo y por inspección llegamos a ⟶ $\boxed{[S_i, S_j] = i\,\varepsilon_{ijk}\,\hbar\, S_k}$

$$\{S_i, S_j\} = S_i S_j + S_i S_j \Rightarrow$$

$$\{S_i, S_j\}_{i=j} = \hbar\left(\frac{\hbar}{2}\mathbb{I}\right) = \frac{\hbar^2}{2}\mathbb{I} \qquad \{S_i, S_j\}_{i\neq j} = 0 \quad \therefore$$

Juntando todo otra vez es ⟶ $\boxed{\{S_i, S_j\} = \frac{\hbar^2}{2}\mathbb{I} = \frac{\hbar^2}{2}\delta_{ij}}$

3.

$$\{|+\rangle, |-\rangle\} \quad \text{con} \quad |+\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} \quad |-\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_y = i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

a)

$$\sigma_x^\dagger \equiv (\sigma^{*t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

$$\sigma_y^\dagger = -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^t = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_y$$

$$\sigma_z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

$\Rightarrow$ **Las matrices son hermíticas**

$$\sigma_x |+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\rangle$$

$$\sigma_x |-\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$$

$$\sigma_y |+\rangle = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|-\rangle$$

$$\sigma_y |-\rangle = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -i|+\rangle$$

$$\sigma_z |+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$$

$$\sigma_z |-\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|-\rangle$$

* Para σ_x

$$\sigma_x |\alpha\rangle = \lambda |\alpha\rangle$$

$$(\sigma_x - \lambda \mathbb{1})|\alpha\rangle = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}|\alpha\rangle = 0$$

$$\det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\boxed{\lambda = \begin{cases} +1 & (1) \\ -1 & (2) \end{cases}}$$

$|\alpha_1\rangle$; $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -a_1 + a_2 = 0$, $a_1 = a_2$ $\quad \boxed{|\alpha_1\rangle = |+\rangle + |-\rangle}$

$|\alpha_2\rangle$; $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad a_1 + a_2 = 0$ $\quad \boxed{|\alpha_2\rangle = |+\rangle - |-\rangle}$

$$\sigma_x |\alpha_2\rangle = |-\rangle - |+\rangle = \underbrace{-1}_{\lambda_2}|\alpha_2\rangle$$

$$\sigma_x |\alpha_1\rangle = |-\rangle + |+\rangle = |+\rangle + |-\rangle = \underbrace{1}_{\lambda_1}|\alpha_1\rangle$$

* Para σ_y

$$\sigma_y |\alpha\rangle = \lambda |\alpha\rangle$$

$$(\sigma_y - \lambda \mathbb{1})|\alpha\rangle = \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix}|\alpha\rangle = 0$$

$$\det\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 + i^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\boxed{\begin{matrix} \lambda_1 = +1 \\ \lambda_2 = -1 \end{matrix}}$$

$|\alpha_1\rangle$; $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -a_1 - i a_2 = 0$, $a_2 = \frac{a_1}{-i} = i a_1$ $\quad \boxed{\alpha_1 = |+\rangle + i|-\rangle}$

$|\alpha_2\rangle$; $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad a_1 - i a_2 = 0$ $\quad \boxed{\alpha_2 = |+\rangle - i|-\rangle}$

$$\sigma_y |\alpha_1\rangle = i|-\rangle + i(-i)|+\rangle = |+\rangle + i|-\rangle = 1 \cdot |\alpha_1\rangle$$

$$\sigma_y |\alpha_2\rangle = i|-\rangle - i(-i)|+\rangle = -|+\rangle + i|-\rangle = -[|+\rangle - i|-\rangle] = -1|\alpha_2\rangle$$

* Para σ_z

$$(\sigma_z - \lambda \mathbb{1}) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = -(1-\lambda)(-1-\lambda) = -[-1+\lambda-\lambda+\lambda^2] = 1-\lambda^2 = 0$$

$$\boxed{\lambda = \begin{cases} +1 \\ -1 \end{cases}}$$

$|\alpha_1\rangle$; $\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -2a_2 = 0 \rightarrow$ $\boxed{\begin{matrix} |\alpha_1\rangle = |+\rangle \\ |\alpha_2\rangle = |-\rangle \end{matrix}}$

$\therefore$ análogamente será $\longrightarrow$

$$\sigma_z |+\rangle = 1|+\rangle \quad , \quad \sigma_z|-\rangle = -|-\rangle$$

b)

$$\det(\sigma_x) = 0-1 = -1$$
$$\det(\sigma_y) = 0-(-i^2) = -1 \quad \Rightarrow \quad \boxed{\det(\sigma_k) = -1}$$
$$\det(\sigma_z) = -1-0 = -1$$

$$tr(\sigma_x) = 0$$
$$tr(\sigma_y) = 0 \quad \Rightarrow \quad \boxed{tr(\sigma_k) = 0}$$
$$tr(\sigma_z) = 1+-1 = 0$$

$$\sigma_x\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\sigma_y\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}-i^2&0\\0&-i^2\end{pmatrix} \quad \Rightarrow \quad \boxed{\sigma_i^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbb{1}}$$
$$\sigma_z\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\sigma_x\sigma_y = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = i\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i.\sigma_z$$
$$\sigma_x\sigma_z = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} = -i\begin{pmatrix}0&-i\\i&\end{pmatrix} = -i.\sigma_y$$
$$\sigma_y\sigma_z = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix} = i\begin{pmatrix}0&1\\1&0\end{pmatrix} = i.\sigma_x$$
$$\sigma_z\sigma_y = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = -i\begin{pmatrix}0&1\\1&0\end{pmatrix} = -i.\sigma_x$$
$$\sigma_z\sigma_x = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = \frac{i}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = -i\begin{pmatrix}0&i\\-i&0\end{pmatrix} = i\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i.\sigma_y$$
$$\sigma_y\sigma_x = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}-i&0\\0&i\end{pmatrix} = -i\begin{pmatrix}1&0\\0&-1\end{pmatrix} = -i.\sigma_z$$

$\Rightarrow$

$$\sigma_j\sigma_k \underset{j=k}{=} \mathbb{1} = \mathbb{1}$$
$$\sigma_j\sigma_k \underset{j\neq k}{=} i.\varepsilon_{jkl}.\sigma_l \quad \Rightarrow \quad \boxed{\sigma_j\sigma_k = i\,\varepsilon_{jkl}.\sigma_l + \mathbb{1}\,\delta_{jk}}$$

c) Se verifica:

$$\sigma_i = \frac{2}{\hbar}S_i \qquad i = x,y,z$$

Las matrices de Pauli son múltiplos de los operadores de spin S_i

4. a)

$\{|a'\rangle, |a''\rangle, |a'''\rangle, \dots, |a^n\rangle\}$ base , conozco $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \dots$
$\langle a'|\beta\rangle, \langle a''|\beta\rangle, \dots$

$$|\alpha\rangle\langle\beta| = \sum_{a'}\sum_{a''} |a''\rangle\underbrace{\langle a''|\alpha\rangle\langle\beta|a'\rangle}_{\text{matriz}}\langle a'|$$

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \langle a^1|\alpha\rangle\langle\beta|a^1\rangle & \langle a^1|\alpha\rangle\langle\beta|a^2\rangle & \cdots\langle a^1|\alpha\rangle\langle\beta|a^N\rangle \\ \langle a^2|\alpha\rangle\langle\beta|a^1\rangle & \langle a^2|\alpha\rangle\langle\beta|a^2\rangle & \vdots \\ \vdots & \vdots & \ddots \\ \langle a^N|\alpha\rangle\langle\beta|a^1\rangle & \cdots\cdots & \langle a^N|\alpha\rangle\langle\beta|a^N\rangle \end{pmatrix}$$

$\sigma_z |+\rangle = 1|+\rangle \quad , \quad \sigma_z|-\rangle = -|-\rangle$

b)

$$\begin{aligned} \det(\sigma_x) &= 0-1 = -1 \\ \det(\sigma_y) &= 0-(-i^2) = -1 \\ \det(\sigma_z) &= -1-0 = -1 \end{aligned} \quad \Rightarrow \quad \boxed{\det(\sigma_k) = -1}$$

$$\begin{aligned} tr(\sigma_x) &= 0 \\ tr(\sigma_y) &= 0 \\ tr(\sigma_z) &= 1+-1 = 0 \end{aligned} \quad \Rightarrow \quad \boxed{tr(\sigma_k) = 0}$$

$$\begin{aligned} \sigma_x\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_y\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} \\ \sigma_z\sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad \Rightarrow \quad \boxed{\sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}}$$

$$\begin{aligned} \sigma_x\sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \cdot \sigma_z \\ \sigma_x\sigma_z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & \end{pmatrix} = -i \cdot \sigma_y \\ \sigma_y\sigma_z &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \cdot \sigma_x \\ \sigma_z\sigma_y &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \cdot \sigma_x \\ \sigma_z\sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{i}{i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \cdot \sigma_y \\ \sigma_y\sigma_x &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i \cdot \sigma_z \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \sigma_j\sigma_k &\underset{j=k}{=} \mathbb{1} = \mathbb{1} \\ \sigma_j\sigma_k &\underset{j\neq k}{=} i \cdot \varepsilon_{jkl} \cdot \sigma_l \end{aligned} \quad \Rightarrow \quad \boxed{\sigma_j\sigma_k = i \cdot \varepsilon_{jkl} \cdot \sigma_l + \mathbb{1}\,\delta_{jk}}$$

c) Se verifica:

$$\sigma_i = \frac{2}{\hbar} S_i \qquad i = x,y,z$$

Las matrices de Pauli son múltiplos de los operadores de spin S_i

4. a)

$\{|a'\rangle, |a''\rangle, |a'''\rangle, \dots, |a^n\rangle\}$ base , conozco $\langle a'|\alpha\rangle, \langle a'|\alpha\rangle, \dots$ $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \dots$

$$|\alpha\rangle\langle\beta| = \sum_{a'}\sum_{a''} |a'\rangle \underbrace{\langle a''|(|\alpha\rangle\langle\beta|)|a'\rangle}_{\text{matriz}} \langle a'|$$

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \langle a^1|\alpha\rangle\langle\beta|a^1\rangle & \langle a^1|\alpha\rangle\langle\beta|a^2\rangle & \cdots & \langle a^1|\alpha\rangle\langle\beta|a^N\rangle \\ \langle a^2|\alpha\rangle\langle\beta|a^1\rangle & \langle a^2|\alpha\rangle\langle\beta|a^2\rangle & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ \langle a^N|\alpha\rangle\langle\beta|a^1\rangle & \cdots & \cdots & \langle a^N|\alpha\rangle\langle\beta|a^N\rangle \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \langle a^1|\alpha\rangle\langle a^1|\beta\rangle^* & \cdots & \langle a^1|\alpha\rangle\langle a^N|\beta\rangle^* \\ \vdots & \ddots & \vdots \\ \langle a^N|\alpha\rangle\langle a^1|\beta\rangle^* & \cdots & \langle a^N|\alpha\rangle\langle a^N|\beta\rangle^* \end{pmatrix}$$

b)

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \langle +|\alpha\rangle\langle\beta|+\rangle & \langle +|\alpha\rangle\langle\beta|-\rangle \\ \langle -|\alpha\rangle\langle\beta|+\rangle & \langle -|\alpha\rangle\langle\beta|-\rangle \end{pmatrix}$$

$|S_x\ \hbar/2\rangle$ → AutoEstado tal que S_x tiene autovalor $\hbar/2$

$S_z|\alpha\rangle = \hbar/2\,|\alpha\rangle$ $\quad |\alpha\rangle = |+\rangle$

$S_x|\beta\rangle = \hbar/2\,|\beta\rangle$ $\quad |\beta\rangle = \frac{|+\rangle}{\sqrt{2}} + \frac{|-\rangle}{\sqrt{2}}$ $\qquad$ base $\{|+\rangle, |-\rangle\}$

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \langle +|\left([|+\rangle][\frac{\langle +|}{\sqrt{2}} + \frac{\langle -|}{\sqrt{2}}]\right)|+\rangle & \langle +|\left([|+\rangle][\frac{\langle +|}{\sqrt{2}} + \frac{\langle -|}{\sqrt{2}}]\right)|-\rangle \\ \langle -|\left([|+\rangle][\frac{\langle +|}{\sqrt{2}} + \frac{\langle -|}{\sqrt{2}}]\right)|+\rangle & \langle -|\left([|+\rangle][\frac{\langle +|}{\sqrt{2}} + \frac{\langle -|}{\sqrt{2}}]\right)|-\rangle \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

5.

$|i\rangle$, $|j\rangle$ autoestados de A hermítico $\Rightarrow$

$A|i\rangle = a_i|i\rangle \quad \wedge \quad A|j\rangle = a_j|j\rangle \qquad a_i, a_j \in \mathbb{C}$

si A es lineal $\Rightarrow \quad A(|i\rangle + |j\rangle) = A|i\rangle + A|j\rangle = a_i|i\rangle + a_j|j\rangle$

Todos los operadores son lineales, a menos que se aclare lo contrario

necesito algo de la pinta $\#(|i\rangle + |j\rangle)$

Pero A es hermítico $\Rightarrow a_i, a_j \in \mathbb{R}$

$= a_i\left(|i\rangle + \frac{a_j}{a_i}|j\rangle\right)$

$\Rightarrow \boxed{a_j = a_i} \Rightarrow |i\rangle$ y $|j\rangle$ son autoestados de A

Será autoestado solo cuando sean $|i\rangle, |j\rangle$ autoestados degenerados correspondientes a un mismo autovalor

6.

$\{|a^i\rangle\}$ autoestados de A ; no hay degeneración

¿ $\prod_{i=1}^{N}(A - a^i)$ es el operador nulo ? $\quad$ es : $(A-a^1)(A-a^2)\ldots(A-a^N)$

$\Rightarrow$ Aplico primeramente sobre un autoket $|a^k\rangle$

$$\prod^{N}(A - a^i\mathbb{1})|a^k\rangle = \prod^{N-1}(A - a^i\mathbb{1})(A - a^N\mathbb{1})|a^k\rangle$$

$$= \prod_{i=1}^{N-1}(A - a^i\mathbb{1})|a^k\rangle(a^k\mathbb{1} - a^N\mathbb{1})$$

$$= \prod_{i=1}^{N-2} (A - a^i \mathbb{1}) |a^k\rangle (a^k - a^{N-1}) \mathbb{1} (a^k - a^N) \mathbb{1}$$

$$\prod_{i=1}^{N} (A - a^i \mathbb{1}) |a^k\rangle = (A - a^1 \mathbb{1})(A - a^2 \mathbb{1}) \ldots (A - a^k \mathbb{1}) |a^k\rangle \ldots (a^k - a^N)$$

en algún momento llegamos al autovalor a^k $\Rightarrow$

$$\prod_{i=1}^{N} (A - a^i \mathbb{1}) |a^k\rangle = (A - a^1 \mathbb{1}) \ldots \underbrace{(a^k - a^k)}_{=0} \mathbb{1} \ldots (a^k - a^N) = \boxed{0}$$

Aplico sobre un ket genérico $|\alpha\rangle = \sum_k c_k |a^k\rangle$ $\Rightarrow$

$$\underbrace{\prod_{i=1}^{N} (A - a^i \mathbb{1})}_{\equiv \mathcal{O}} \left(\sum_k c_k |a^k\rangle \right) = \mathcal{O} |\alpha\rangle$$

$$\mathcal{O} \left(c_1 |a^1\rangle + c_2 |a^2\rangle + \ldots + c_N |a^N\rangle \right) =$$

en cada término tenemos un factor $\begin{cases} A - a^1 \mathbb{1} \\ A - a^2 \mathbb{1} \\ \vdots \\ A - a^N \mathbb{1} \end{cases}$

$\boxed{\prod_{i=1}^{N} (A - a^i \mathbb{1}) \text{ es el operador nulo}}$ $\Leftarrow$ Se anulan todos los términos

b)

$$\prod_{a'' \neq a'} \frac{(A - a'')}{(a' - a'')} \qquad \frac{(A - a^1 \mathbb{1})(A - a^2 \mathbb{1})(A - a^3 \mathbb{1}) \ldots (A - a^N \mathbb{1})}{(a' - a^1)(a' - a^2) \ldots (a' - a^N)}$$

$$\prod_{i \neq l} \frac{(A - a^i \mathbb{1})}{(a^l - a^i)} |a^k\rangle = 0 \longrightarrow \text{pues cuando } k = i \rightarrow k \neq l \quad \frac{\ldots |a^k\rangle (a^k - a^k) \ldots}{\ldots (a^l - a^k) \ldots}$$

$$= 1 \longrightarrow \text{pues si } k = l,\ i \neq l \quad \frac{|a^l\rangle (a^l - a^i)}{(a^l - a^i)}$$

$$\frac{|a^l\rangle (a^l - a^1)(a^l - a^2) \ldots (a^l - a^N)}{(a^l - a^1)(a^l - a^2) \ldots (a^l - a^N)}$$

No hay término $i = l$

$$\ldots (a^l - a^{(l-1)})(a^l - a^{(l+1)}) \ldots$$

$\Rightarrow$ juntando todo

$$\boxed{\prod_{a'' \neq a'}^{N} \frac{(A - a'' \mathbb{1})}{(a' - a'')} = \delta_{a'a''}}$$

c)

$$A = S_z = \frac{\hbar}{2} \left(|+\rangle\langle+| - |-\rangle\langle-| \right)$$

$$S_z |+\rangle = \frac{\hbar}{2} |+\rangle \qquad S_z |-\rangle = -\frac{\hbar}{2} |-\rangle$$

$$\mathcal{O} = \left(S_z - \frac{\hbar}{2} \mathbb{1} \right) \left(S_z + \frac{\hbar}{2} \mathbb{1} \right)$$

$|\alpha\rangle$ genérico

$$\mathcal{O} |\alpha\rangle = \frac{\hbar^2}{2^2} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} =$$

$$\mathcal{O} |\alpha\rangle = \left(\frac{\hbar}{2} \right)^2 \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} = \boxed{0}$$

; operador nulo, como lo probábamos

$$\frac{(S_z + \overbrace{\hbar/2}^{a^1} \mathbb{1})}{(\underbrace{\hbar/2}_{a^1} - \underbrace{(-\hbar/2)}_{a^2})} \underbrace{|+\rangle}_{a^1} = \frac{(\hbar/2 + \hbar/2) \mathbb{1}}{(\hbar/2 + \hbar/2)} = \boxed{\mathbb{1}}$$

pues $k = l$, $i \neq l$

7. $H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$ $N=2$

$\{|1\rangle, |2\rangle\}$ una base, pero no son autokets

$H|1\rangle = a\left(|1\rangle + |2\rangle\right)$

$H|2\rangle = a\left(-|2\rangle + |1\rangle\right)$

$$H|\gamma\rangle = \gamma|\gamma\rangle \rightarrow (H - \gamma \mathbb{1})|\gamma\rangle = 0$$

Hay que escribir la representación matricial de H en la base $\{|1\rangle, |2\rangle\}$

$$H \doteq \begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle \end{pmatrix}$$

$$H \doteq \begin{pmatrix} \langle 1|\left(a|1\rangle + a|2\rangle\right) & \langle 1|\left(a|1\rangle - a|2\rangle\right) \\ \langle 2|\left(a|1\rangle + a|2\rangle\right) & \langle 2|\left(a|1\rangle - a|2\rangle\right) \end{pmatrix}$$

$$H \doteq \begin{pmatrix} a & a \\ a & -a \end{pmatrix} \Rightarrow (H - \gamma\mathbb{1}) = \begin{pmatrix} a-\gamma & a \\ a & -a-\gamma \end{pmatrix}$$

$$\begin{vmatrix} a-\gamma & a \\ a & -a-\gamma \end{vmatrix} = -(a-\gamma)(a+\gamma) - a^2 = -a^2 + \gamma^2 - a^2 = -2a^2 + \gamma^2 = 0$$

autovalores de energía → $\boxed{\begin{matrix}\gamma_1 = \sqrt{2}\,a \\ \gamma_2 = -\sqrt{2}\,a\end{matrix}}$

* γ^1

$$\begin{pmatrix} a - \sqrt{2}a & a \\ a & -a - \sqrt{2}a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow a(1-\sqrt{2})x_1 + a x_2 = 0$$

$$x_2 = (\sqrt{2}-1)x_1$$

$$V_1 = x_1 \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$$x_1^2\left(1 + (\sqrt{2}-1)^2\right) = 1$$

$$x_1^2\left(1 + (2 - 2\sqrt{2} + 1)\right) = 1$$

$$x_1^2(4 - 2\sqrt{2}) = 1$$

$$x_1 = \sqrt{\frac{1}{2(2-\sqrt{2})}} = \frac{1}{\sqrt{2(2-\sqrt{2})}}$$

$$V_1 = \begin{pmatrix} \frac{1}{\sqrt{2(2-\sqrt{2})}} \\ -\frac{1-\sqrt{2}}{\sqrt{2(2-\sqrt{2})}} \end{pmatrix}$$

* γ^2

$$\begin{pmatrix} a + \sqrt{2}a & a \\ a & -a + \sqrt{2}a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1+\sqrt{2})x_1 + x_2 = 0 \qquad V_2 = x_1\begin{pmatrix} 1 \\ -(1+\sqrt{2}) \end{pmatrix}$$

$$x_1^2\left[1 + (1+\sqrt{2})^2\right] = 0$$

$$x_1^2(1 + 1 + 2 + 2\sqrt{2}) = 0$$

$$x_1^2 \cdot 2(2+\sqrt{2}) = 0$$

$$x_1 = \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}$$

$$V_2 = \begin{pmatrix} \frac{1}{\sqrt{2(2-\sqrt{2})}} \\ \frac{-(1+\sqrt{2})}{\sqrt{2(2-\sqrt{2})}} \end{pmatrix}$$

$$|V_1\rangle = \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2}}}|1\rangle - \frac{1-\sqrt{2}}{\sqrt{2}\sqrt{2-\sqrt{2}}}|2\rangle$$

$$|V_2\rangle = \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2}}}|1\rangle - \frac{1+\sqrt{2}}{\sqrt{2}\sqrt{2-\sqrt{2}}}|2\rangle$$

$$\sqrt{2}\sqrt{2-\sqrt{2}}$$
$$\sqrt{2\cdot 2\left(1-\frac{1}{\sqrt{2}}\right)}$$
$$2\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

Comprobemos uno para "peace of mind"

$$H|V_2\rangle = a\frac{1}{\sqrt{2(2-\sqrt{2})}}(|2\rangle + |1\rangle) - a\left(\frac{1-\sqrt{2}}{\sqrt{2(2-\sqrt{2})}}(|1\rangle - |2\rangle)\right) \rightarrow = \frac{a}{\sqrt{\;}} + \frac{a\sqrt{2}}{\sqrt{\;}}$$

$$\frac{a}{\sqrt{\;}}|2\rangle + \frac{a}{\sqrt{\;}}|1\rangle - \frac{a}{\sqrt{\;}}|1\rangle + \frac{a}{\sqrt{\;}}|2\rangle + \frac{a\sqrt{2}}{\sqrt{\;}}|1\rangle - \frac{a\sqrt{2}}{\sqrt{\;}}|2\rangle$$

$$\frac{2a - a\sqrt{2}}{\sqrt{\;}}|2\rangle + \frac{a\sqrt{2}}{\sqrt{\;}}|1\rangle = \sqrt{2}\,a\left(\frac{|1\rangle}{\sqrt{\;}} + \frac{\sqrt{2}-1}{\sqrt{\;}}|2\rangle\right)$$

$$H|V_2\rangle = \sqrt{2}a\left(\frac{1}{\sqrt{2(2-\sqrt{2})}}|1\rangle - \frac{1-\sqrt{2}}{\sqrt{2(2-\sqrt{2})}}|2\rangle\right) = \sqrt{2}a|V_2\rangle$$

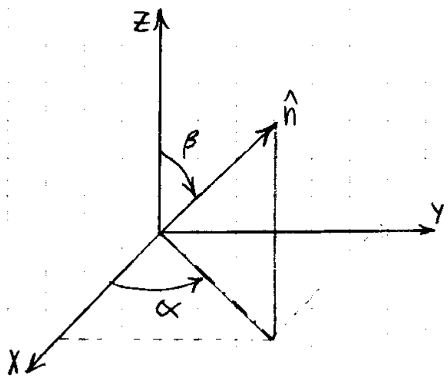
Los autovalores correspondientes son

$\sqrt{2}a \rightarrow V_2$
$-\sqrt{2}a \rightarrow V_1$

8. $|\vec{S}.\hat{n}\,;+\rangle$ Ket estado tal que: $\{|+\rangle, |-\rangle\}$ base

$$\underset{\text{(operador)}}{\vec{S}.\hat{n}}\,|\vec{S}.\hat{n}\,;+\rangle = \frac{\hbar}{2}|\vec{S}.\hat{n}\,;+\rangle$$

es decir que $\frac{\hbar}{2}$ sea autovalor de $\vec{S}.\hat{n}$ (operador) aplicado a nuestro ket estado



Denotamos $\vec{S}.\hat{n} \equiv S_{\hat{n}}$ $\Rightarrow$ hay que construir un ket $|S_{\hat{n}};+\rangle$ que sea autoket del operador $S_{\hat{n}}$ con autovalor $\hbar/2$

$$\left(S_{\hat{n}} - \frac{\hbar}{2}\mathbb{1}\right)|S_{\hat{n}};+\rangle = 0$$

$$\hat{n} = \cos\beta.\hat{z} + \text{sen}\beta.\cos\alpha\,\hat{x} + \text{sen}\beta.\text{sen}\alpha\,\hat{y}$$

$$\vec{S}.\hat{n} = \cos\beta.\vec{S}.\hat{z} + \text{sen}\beta.\cos\alpha\,\vec{S}.\hat{x} + \text{sen}\beta.\text{sen}\alpha\,\vec{S}.\hat{y}$$

Supongamos $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Usando lo que se hizo en ejercicio 2 tomamos:

$$S_{\hat{n}}|+\rangle = \cos\beta\, S_{\hat{z}}|+\rangle + \text{sen}\beta.\left[\cos\alpha\, S_{\hat{x}}|+\rangle + \text{sen}\alpha\, S_{\hat{y}}|+\rangle\right]$$

$$= \frac{\hbar}{2}\cos\beta|+\rangle + \text{sen}\beta.\left[\cos\alpha\frac{\hbar}{2}|-\rangle + \text{sen}\alpha\, i\frac{\hbar}{2}|-\rangle\right]$$

$$S_{\hat{n}}|+\rangle = \frac{\hbar}{2}\left[\cos\beta|+\rangle + \text{sen}\beta\,.\,e^{i\alpha}|-\rangle\right] \qquad [A]$$

$$S_{\hat{n}}|-\rangle = \cos\beta\, S_{z}|-\rangle + \text{sen}\beta\left[\cos\alpha\, S_{\hat{x}}|-\rangle + \text{sen}\alpha\, S_{\hat{y}}|-\rangle\right]$$

$$-\frac{\hbar}{2}\cos\beta\,|-\rangle + \text{sen}\beta\left[\cos\alpha\frac{\hbar}{2}|+\rangle - \text{sen}\alpha\frac{i\hbar}{2}|+\rangle\right]$$

$$S_{\hat{n}}|-\rangle = \frac{\hbar}{2}\left[-\cos\beta|-\rangle + \text{sen}\beta\,.\,e^{-i\alpha}|+\rangle\right]$$

$$S_{\hat{n}} \doteq \begin{pmatrix}\langle +|S_{\hat{n}}|+\rangle & \langle +|S_{\hat{n}}|-\rangle \\ \langle -|S_{\hat{n}}|+\rangle & \langle -|S_{\hat{n}}|-\rangle\end{pmatrix}$$

NO CONFUNDIR

en [A] tenemos:

$$\vec{S}.\hat{n}\,|\lambda\rangle = \frac{\hbar}{2}|\delta\rangle$$

pero $|\lambda\rangle = |+\rangle \neq |\delta\rangle$

$\Rightarrow |\delta\rangle$ no es el autoket buscado

Representación matricial de $S_{\hat{n}}$ →

$$S_{\hat{n}} \doteq \begin{pmatrix}\frac{\hbar}{2}\cos\beta & \frac{\hbar}{2}\text{sen}\beta.e^{-i\alpha} \\ \frac{\hbar}{2}\text{sen}\beta\, e^{i\alpha} & -\frac{\hbar}{2}\cos\beta\end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\beta & \text{sen}\beta.e^{-i\alpha} \\ \text{sen}\beta.e^{i\alpha} & -\cos\beta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow$$

$$\cos\beta.a + \text{sen}\beta.e^{-i\alpha}.b = a$$
$$a.\text{sen}\beta.e^{i\alpha} - \cos\beta.b = b$$

$$b(1+\cos\beta) = a.\text{sen}\beta.e^{i\alpha}$$

$$b = a\frac{[1-\cos\beta]}{\text{sen}\beta}e^{i\alpha}$$

$$b = a.\frac{2\,\text{sen}^2(\beta/2)}{\text{sen}[2(\beta/2)]}.e^{i\alpha}$$

$$b = a\frac{\cancel{2}\,\cancel{\text{sen}\,\beta/2}\,.\,\text{sen}\,\beta/2}{\cancel{2}\,.\,\cancel{\text{sen}\,\frac{\beta}{2}}\,.\cos\frac{\beta}{2}}\,.\,e^{i\alpha} = a.e^{i\alpha}\tan\left(\frac{\beta}{2}\right)$$

$$b = a.\frac{\text{sen}\,\beta}{(1+\cos\beta)}.\,e^{i\alpha} = \frac{a.\cancel{2}\,\text{sen}\,\beta/2\,\cancel{\cos\beta/2}}{\cancel{2}\cos^2\beta/2}\,e^{i\alpha} = a\,e^{i\alpha}\tan\left(\frac{\beta}{2}\right)$$

$$|a|^2+|b|^2 = 1 = a^2 + a^2\tan^2\left(\frac{\beta}{2}\right) \rightarrow a^2 = \frac{1}{1+\tan^2\left(\frac{\beta}{2}\right)} = \frac{1}{1+\frac{\text{sen}^2(\beta/2)}{\cos^2(\beta/2)}} = \frac{\cos^2(\beta/2)}{1}$$

$$b = \cancel{\cos\beta/2}\,e^{i\alpha}.\frac{\text{sen}\,\beta/2}{\cancel{\cos\beta/2}} \Rightarrow \boxed{|\vec{S}.\hat{n};+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \text{sen}\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle}$$

CA

$$\frac{1-\cos 2A}{2} = \text{sen}^2 A$$

$$\frac{1+\cos 2A}{2} = \cos^2 A$$

* <u>otro modo</u>

Podríamos haber resuelto el problema de autovalores como sigue:

$$\begin{vmatrix} \frac{\hbar}{2}\cos\beta-\lambda & \frac{\hbar}{2}\text{sen}\,\beta\,e^{-i\alpha} \\ \frac{\hbar}{2}\text{sen}\,\beta\,e^{i\alpha} & -\frac{\hbar}{2}\cos\beta-\lambda \end{vmatrix} = \nearrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2\cos^2\beta + \left(\frac{\hbar}{2}\right)^2\text{sen}^2\beta$$

$$\lambda = +\frac{\hbar}{2}$$

$$\lambda = -\frac{\hbar}{2}$$

$$\left(\frac{\hbar}{2}\cos\beta - \frac{\hbar}{2}\right)x_1 + \left(\frac{\hbar}{2}\,\text{sen}\,\beta.\,e^{-i\alpha}\right)x_2 = 0$$

$$x_2 = \frac{\cancel{\left(\frac{\hbar}{2}\right)}(\cos\beta+1)}{\cancel{\left(\frac{\hbar}{2}\right)}\text{sen}\,\beta.\,e^{-i\alpha}}\,x_1$$

$$|\lambda\rangle = x_1\begin{pmatrix} 1 \\ \left(\frac{1-\cos\beta}{\text{sen}\,\beta}\right)e^{i\alpha}\end{pmatrix}$$

$$x_2 = \left(\frac{1-\cos\beta}{\text{sen}\,\beta}\right)e^{i\alpha}x_1$$

$$= x_1\begin{pmatrix} 1 \\ \frac{\cancel{2}\,\text{sen}^{\cancel{2}}\beta/2}{\cancel{2}\,\cancel{\text{sen}\,\beta/2}\cos\beta/2}.\,e^{i\alpha}\end{pmatrix} = x_1\begin{pmatrix} 1 \\ \tan(\beta/2)e^{i\alpha}\end{pmatrix} \Longrightarrow |x_1|^2 + |x_1|^2.\tan^2\left(\frac{\beta}{2}\right) = 1$$

$$|x_1|^2 = \cos^2(\beta/2)$$

$$|\lambda\rangle = \begin{pmatrix} \cos\beta/2 \\ \cos\left(\frac{\beta}{2}\right).\frac{2.\text{sen}^2\left(\frac{\beta}{2}\right)}{\text{sen}\,\beta}.\,e^{i\alpha}\end{pmatrix}$$

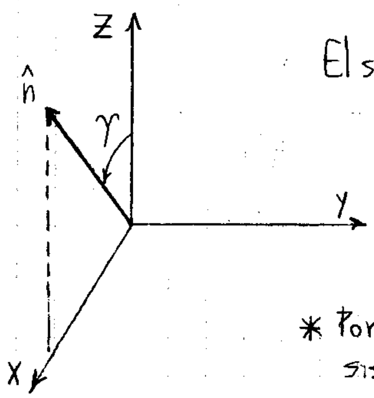
$$\cos\left(\frac{\beta}{2}\right)|+\rangle + \cancel{\cos\left(\frac{\beta}{2}\right)}.\,\cancel{2}\,\text{sen}^{\cancel{2}}\left(\frac{\beta}{2}\right).\,e^{i\alpha}.\frac{1}{\cancel{2}\,\cancel{\text{sen}\left(\frac{\beta}{2}\right)}\,\cancel{\cos\left(\frac{\beta}{2}\right)}}|-\rangle$$

$$\boxed{|\lambda\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \text{sen}\left(\frac{\beta}{2}\right).\,e^{i\alpha}|-\rangle}$$

9.

$\vec{S}.\hat{n}$ operador — autoket con autovalor $\frac{\hbar}{2}$; es un autoestado de $\vec{S}.\hat{n}$

$$\vec{S}.\hat{n}\,|\gamma\rangle = \frac{\hbar}{2}|\gamma\rangle$$



El sistema se halla en $|S.\hat{n};\gamma\rangle$

a)

$$P\left(S_x,\frac{\hbar}{2}\right) = |\langle S_x;+|\vec{S}.\hat{n};+\rangle|^2$$

(estado, estado; normalizado)

* Por el ejercicio 8 el estado en el cual se halla el sistema es:

$$|\vec{S}.\hat{n};+\rangle = |\gamma\rangle = \cos\left(\frac{\gamma}{2}\right)|+\rangle + \text{sen}\left(\frac{\gamma}{2}\right)|-\rangle$$

pues su autovalor era $\hbar/2$. Para ver la probabilidad de obtener $+\hbar/2$ al medir S_x entonces necesito contemplar el estado general $|S_x;+\rangle$

$$\langle S_x;+| = +\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|$$

Probabilidad de $\hbar/2$ en S_x de obtener un autovalor $\hbar/2$ significa que el sistema debe hallarse en $|S_x;+\rangle$ pues $S_x|S_x;+\rangle = \frac{\hbar}{2}|S_x;+\rangle$

$$\langle S_x;+|\vec{S}.\hat{n};+\rangle = \left(\frac{\langle +|}{\sqrt{2}} + \frac{\langle -|}{\sqrt{2}}\right)\left(\cos\left(\frac{\gamma}{2}\right)|+\rangle + \text{sen}\left(\frac{\gamma}{2}\right)|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\cos\left(\frac{\gamma}{2}\right) + \frac{1}{\sqrt{2}}\,\text{sen}\left(\frac{\gamma}{2}\right) \Rightarrow$$

$$|\langle S_x;+|\vec{S}.\hat{n};+\rangle|^2 = \left(\frac{1}{\sqrt{2}}\right)^2\cos^2\left(\frac{\gamma}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)^2\text{sen}^2\left(\frac{\gamma}{2}\right) + \frac{2}{2}\cos\left(\frac{\gamma}{2}\right)\text{sen}\left(\frac{\gamma}{2}\right)$$

$$= \frac{1}{2} + \cos\left(\frac{\gamma}{2}\right)\cdot\text{sen}\left(\frac{\gamma}{2}\right)$$

$$= \frac{1}{2}\left[1 + 2\cos\left(\frac{\gamma}{2}\right)\text{sen}\left(\frac{\gamma}{2}\right)\right]$$

$$\boxed{P\left(\frac{\hbar}{2}\right) = \frac{1}{2}\left[1 + \text{sen}(\gamma)\right]}$$

b)

$$\langle\{S_x - \langle S_x\rangle\}^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right) \Rightarrow$$

$$S_x^2 = \frac{\hbar^2}{2^2}\left(\underbrace{|+\rangle\langle -|+\rangle\langle -|}_{=0} + |-\rangle\langle +|+\rangle\langle -| + |+\rangle\langle -|-\rangle\langle +| + \underbrace{|-\rangle\langle +|-\rangle\langle +|}_{=0}\right)$$

$$S_x^2 = \frac{\hbar^2}{4}\underbrace{\left(|-\rangle\langle -| + |+\rangle\langle +|\right)}_{\sum_{a'}\Lambda_{a'} = \mathbb{1}} = \frac{\hbar^2}{4}\mathbb{1} \Rightarrow$$

$$\langle S_x^2 \rangle = \langle \gamma | S_x^2 | \gamma \rangle = \left(\cos\left(\frac{\gamma}{2}\right)\langle + | + \text{sen}\left(\frac{\gamma}{2}\right)\langle - |\right)\left(\frac{\hbar^2}{4}\mathbb{1}\right)\left(\cos\left(\frac{\gamma}{2}\right)|+\rangle + \text{sen}\left(\frac{\gamma}{2}\right)|-\rangle\right)$$

Valor medio de S_x^2 respecto al estado $|\gamma\rangle$ en el cual se halla

$$\left(\qquad\right)\left(\frac{\hbar^2}{4}\cos\left(\frac{\gamma}{2}\right)|+\rangle + \frac{\hbar^2}{4}\,\text{sen}\left(\frac{\gamma}{2}\right)|-\rangle\right)$$

$$\langle S_x^2 \rangle = \cos^2\left(\frac{\gamma}{2}\right)\frac{\hbar^2}{4} + \text{sen}^2\left(\frac{\gamma}{2}\right)\frac{\hbar^2}{4} = \frac{\hbar^2}{4}$$

$$\langle S_x \rangle = \langle \gamma | S_x | \gamma \rangle$$

$$= \left(\cos\left(\frac{\gamma}{2}\right)\langle + | + \text{sen}\left(\frac{\gamma}{2}\right)\langle - |\right)\frac{\hbar}{2}\left(|+\rangle\langle - | + |-\rangle\langle + |\right)\left(\cos\left(\frac{\gamma}{2}\right)|+\rangle + \text{sen}\left(\frac{\gamma}{2}\right)|-\rangle\right)$$

$$\frac{\hbar}{2}\left(\cos\left(\frac{\gamma}{2}\right)\langle + | + \text{sen}\left(\frac{\gamma}{2}\right)\langle - |\right)\left(\text{sen}\left(\frac{\gamma}{2}\right)|+\rangle + \cos\left(\frac{\gamma}{2}\right)|-\rangle\right)$$

$$\langle S_x \rangle = \frac{\hbar}{2}\left[\text{sen}\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right).2\right] = \frac{\hbar}{2}.\text{sen}\,\gamma$$

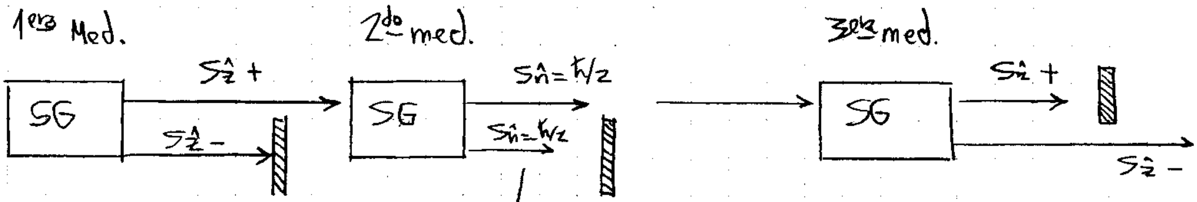
$$(\Delta S_x)^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}.\text{sen}^2\gamma$$

$$\boxed{(\Delta S_x)^2 = \frac{\hbar^2}{4}\left[1 - \text{sen}^2\gamma\right] = \frac{\hbar^2}{4}.\cos^2\gamma}$$

casos particulares

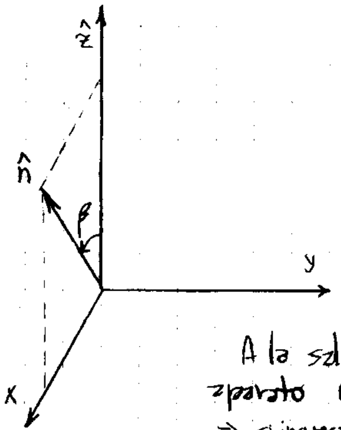
$\gamma = 0$	$(\Delta S_x)^2 = \frac{\hbar^2}{4}$
$\gamma = \frac{\pi}{2}$	$(\Delta S_x)^2 = 0$
$\gamma = \pi$	$(\Delta S_x)^2 = \frac{\hbar^2}{4}$

10. Medición $\equiv$ filtrado selectivo



$\hbar/2$ autovalor de $\vec{S}\cdot\hat{n}$

$S_z = \hbar/2$ → este es el autovalor corresp. al autovector $|+\rangle$



estado

$$S_{\hat z},+ = |S_{\hat z};+\rangle \equiv |+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$$ → ya está normalizado

$$S_{\hat n} \doteq \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \mathrm{sen}\,\beta\\ \mathrm{sen}\,\beta & -\cos\beta\end{pmatrix}$$

A la salida del 2do aparato recogeré $|S_{\hat n},+\rangle$ → estado que corresponde a $\frac{\hbar}{2}$

⇒ si ingresan $|+\rangle$ necesito saber que intensidad tendré

del ej. 8 sabemos que $|S_{\hat n},+\rangle$ es

$$|S_{\hat n},+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \mathrm{sen}\left(\frac{\beta}{2}\right)|-\rangle$$

$$\mathrm{Prob}_{\left(+\frac{\hbar}{2}\right)} = |\langle +|S_{\hat n},+\rangle|^2$$

Intensidad del haz de salida $= \cos^2\left(\frac{\beta}{2}\right)$

Nota

$S_z|+\rangle = +\frac{\hbar}{2}|+\rangle$

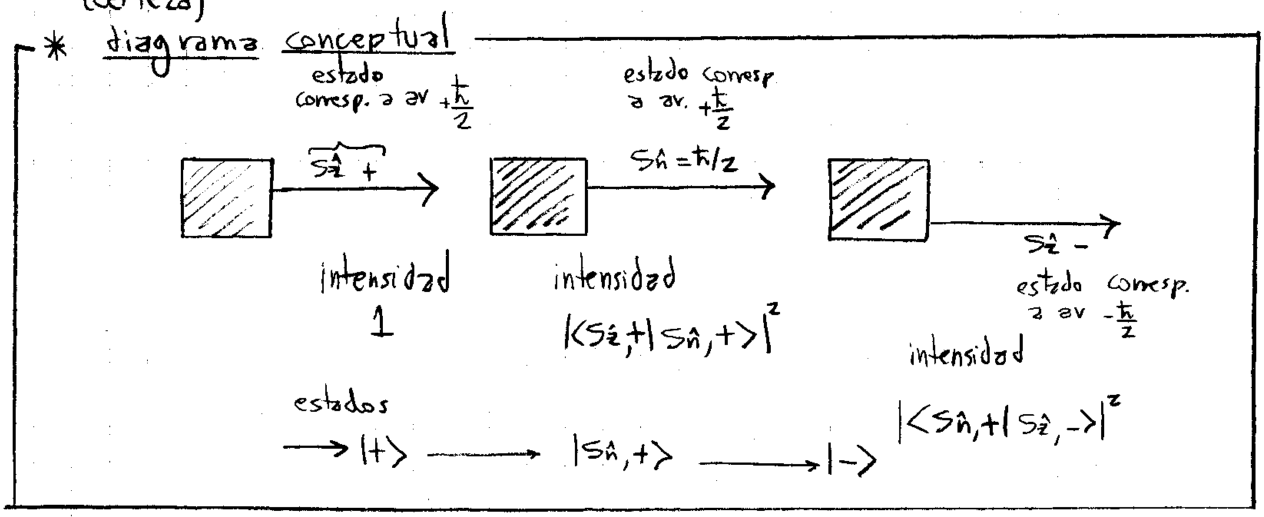
$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$

La 3era medida retiene $-\hbar/2$ (av. corresp. a $|-\rangle$)

⇒ $$\mathrm{Prob}\left(-\hbar/2\right) = |\langle -|S_{\hat n};+\rangle|^2 = \mathrm{sen}^2\left(\frac{\beta}{2}\right)$$ ← intensidad del 3er haz

$$\text{Intensidad final} = \cos^2\left(\frac{\beta}{2}\right)\cdot \mathrm{sen}^2\left(\frac{\beta}{2}\right)$$

* La intensidad del haz final es $(\mathrm{sen}^2\beta)$
* Se lo debe orientar con $\hat n = \hat x$, es decir $\beta = \frac{\pi}{2}$ así es 1 la prob. final (certeza)
* diagrama conceptual



11.

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow$$ desde el vamos tiene $\det\{A\}=0$, pues una fila no es independiente

a) $(\hat{A} - \lambda \mathbb{1})|\alpha\rangle = 0 \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} \rightarrow |A| = (-\lambda)^3 - (-\lambda) - (-\lambda)$

$= -\lambda^3 + 2\lambda = 0$

autovalores $\rightsquigarrow$ $\boxed{\lambda = 0,\ \lambda = +\sqrt{2},\ \lambda = -\sqrt{2}}$ $\quad -\lambda^2 + 2 = 0$, $\lambda = \pm\sqrt{2}$

$\underline{\lambda = 0}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0 \qquad \vec{V} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\alpha_1 \rightarrow$$

$\alpha_2 = 0$

$\alpha_1 = -\alpha_3$

$\alpha_1^2(1+1) = 1$

$\alpha = \frac{1}{\sqrt{2}}$

$$\boxed{V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}} \quad \lambda = 0$$

$\underline{\lambda = -\sqrt{2}}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0 \qquad \vec{V} = \alpha_1\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$\sqrt{2}\alpha_1 + \alpha_2 = 0$

$\alpha_2 = -\sqrt{2}\alpha_1$

$\alpha_2 = -\sqrt{2}\alpha_3$

$\downarrow$

$\alpha_3 = \alpha_1$

$\alpha_1^2(1 + 2 + 1) = 1$

$\alpha_1 = \frac{1}{2}$

$$\boxed{\vec{V}_2 = \begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix}} \quad \lambda = -\sqrt{2}$$

$\underline{\lambda_2 = +\sqrt{2}}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0 \qquad \vec{V} = \alpha_1\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \searrow \alpha_1 = 1/2 \nearrow$$

$$\boxed{\vec{V}_2 = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix}} \quad \lambda = \sqrt{2}$$

$-\sqrt{2}\alpha_1 + \alpha_2 = 0$

$\alpha_2 = \alpha_1\sqrt{2}$

$\alpha_2 - \sqrt{2}\alpha_3 = 0$

$\alpha_2 = \alpha_3\sqrt{2}$ $\nearrow \alpha_1 = \alpha_3$

La normalización es similar a la del caso anterior $\Rightarrow$

- No hay degeneración porque los autovalores no se repiten; lo que sí pasa es que tenemos un autovalor nulo

b)

- Se verá luego que esta matriz, amén de un $\hbar$, es la correspondiente al momento angular L_x y conllevа la posibilidad de tener valor nulo, es decir, que no haya proyección del $\vec{L}$ en $\hat{x}$

12.

A, B observables

$\{|\alpha'\rangle\}$ autokets simultáneos de A y B → conjunto completo ortonormal

$A|\alpha'\rangle = a'|\alpha'\rangle$
$B|\alpha'\rangle = b'|\alpha'\rangle$

$|\Psi\rangle = \sum_{\alpha'} c_{\alpha'}|\alpha'\rangle$ ← cualquier ket

$A.B|\Psi\rangle - BA|\Psi\rangle =$

$A.B\sum_{\alpha'} c_{\alpha'}|\alpha'\rangle - BA\sum_{\alpha'} c_{\alpha'}|\alpha'\rangle =$

$\sum_{\alpha'} A c_{\alpha'} b'|\alpha'\rangle - \sum_{\alpha'} B c_{\alpha'} a'|\alpha'\rangle =$

$\sum_{\alpha'} c_{\alpha'} b'a'|\alpha'\rangle - \sum_{\alpha'} c_{\alpha'} a'b'|\alpha'\rangle =$

$b'a'\sum_{\alpha'}|\alpha'\rangle\langle\alpha'|\Psi\rangle - b'a'\sum_{\alpha'}|\alpha'\rangle\langle\alpha'|\Psi\rangle = 0$

$b'a'\left(\sum_{\alpha'}|\alpha'\rangle\langle\alpha'|\Psi\rangle - \sum_{\alpha}|\alpha'\rangle\langle\alpha'|\Psi\rangle\right) = 0$

$\Rightarrow$ $[A,B] = 0$ si $b', a' \neq 0$

si a' ó b' son nulos no se sabe

13.

$\{A,B\} = 0 \rightarrow AB + BA = 0$ A,B hermíticos

Si $\exists$ autoket común de A y B $\Rightarrow$ será $|\alpha\rangle$ tal que:

$A|\alpha\rangle = a|\alpha\rangle \wedge B|\alpha\rangle = b|\alpha\rangle \Rightarrow$

$AB|\alpha\rangle + BA|\alpha\rangle = Ab|\alpha\rangle + Ba|\alpha\rangle =$

$b.a|\alpha\rangle + a.b|\alpha\rangle = (b.a + ab)|\alpha\rangle = 0$

$\Rightarrow$ $2ba \neq 0 \Rightarrow$ No es posible $|\alpha\rangle$

$b, a = 0 \Rightarrow$ $|\alpha\rangle$ es posible

14.

A_1, A_2 observables que no involucran t explícitamente

$[A_1, A_2] \neq 0$; $[A_1, H] = 0$; $[A_2, H] = 0$

$H|e'\rangle = e'|e'\rangle$ ⇝ autoestados de energía

$HA_2|e'\rangle = e'A_2|e'\rangle$; $HA_1|e'\rangle = e'A_1|e'\rangle$

$HA_1A_2|e'\rangle = e'A_1A_2|e'\rangle$; $HA_2A_1|e'\rangle = e'A_2A_1|e'\rangle$

$\equiv H(|\gamma'\rangle) = e'(|\gamma'\rangle)$; $\equiv H(|\gamma''\rangle) = e'(|\gamma''\rangle)$

Entonces los autoestados de energía son en general degenerados porque como:

$$A_1 A_2 \neq A_2 A_1 \Rightarrow A_1 A_2 |e'\rangle \neq A_2 A_1 |e'\rangle \Rightarrow$$

se tienen dos autoestados de H que son $|\gamma'\rangle$ y $|\gamma''\rangle$ que verifican → $|\gamma'\rangle \neq |\gamma''\rangle$ pero tienen el mismo autovalor e'

∴ **Los autoestados de energía son, en general, degenerados**

15.

a)

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0 \rightarrow (|\alpha\rangle + \lambda|\beta\rangle)^2 \geq 0$$

sea $|\gamma\rangle = |\alpha\rangle + \lambda|\beta\rangle \Rightarrow$

$\updownarrow$ DC

$\langle\gamma| = \langle\alpha| + \lambda^*\langle\beta| \Rightarrow$

$\Big\}\ \langle\gamma|\gamma\rangle \geq 0$ ← por propiedades del valor absoluto

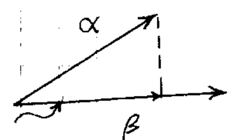
Vale

si elegimos $\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$ para que valga la desigualdad de Schwarz

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

$$\left[\langle\alpha| - \frac{\langle\alpha|\beta\rangle\langle\beta|}{\langle\beta|\beta\rangle}\right]\left[|\alpha\rangle - \frac{|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\right] \geq 0$$

$$\langle\alpha|\alpha\rangle - \frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} + \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$$



$|\alpha|^2 \cdot |\beta|^2 \geq |\alpha \cdot \beta|^2$

Desigualdad de Schwartz para estados

$$\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\beta|\alpha\rangle|^2}$$

b) A, B observables → $A^* = A$, $B = B^*$, $\langle A\rangle = \langle A\rangle^*$, $\langle B\rangle = \langle B\rangle^*$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2 \quad \text{con } \Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle \mathbb{1}$$

→ operador

← estado cualquiera

$$\gamma \equiv \Delta A|\alpha\rangle = \hat{A}|\alpha\rangle + \langle\hat{A}\rangle\mathbb{1}|\alpha\rangle$$

$$\gamma \equiv \Delta B|\alpha\rangle = \hat{B}|\alpha\rangle + \langle\hat{B}\rangle \mathbb{1}|\alpha\rangle \Rightarrow$$

$$\langle\gamma|\gamma\rangle\langle\gamma|\gamma\rangle \geq |\langle\gamma|\gamma\rangle|^2$$ ∴ usando lo anterior

$$\langle\alpha|(\mathbb{1}\langle\hat{A}\rangle + \hat{A})(\hat{A} + \langle\hat{A}\rangle\mathbb{1})|\alpha\rangle\langle\alpha|(\mathbb{1}\langle\hat{B}\rangle + \hat{B})(\hat{B} + \mathbb{1}\langle\hat{B}\rangle)|\alpha\rangle$$
$$\geq |\langle\alpha|(\hat{A} + \langle\hat{A}\rangle\mathbb{1})(\hat{B} + \langle\hat{B}\rangle\mathbb{1})|\alpha\rangle|^2$$

$$\langle\alpha|(\Delta A)^2|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle \geq |\langle\alpha|\Delta A\,\Delta B|\alpha\rangle|^2$$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\,\Delta B\rangle|^2$$

$$\Delta A\,\Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}$$

$$\langle\Delta A\,\Delta B\rangle = \frac{1}{2}\langle[\Delta A, \Delta B]\rangle + \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle$$

$\langle[\Delta A, \Delta B]\rangle \in \mathfrak{Im}$; $\langle\{\Delta A, \Delta B\}\rangle \in \mathbb{R}$ pues es $\{\Delta A, \Delta B\}$ hermítico

$$[\Delta A, \Delta B] = (\hat{A} - \langle\hat{A}\rangle\mathbb{1})(\hat{B} - \langle\hat{B}\rangle\mathbb{1}) - (\hat{B} - \langle\hat{B}\rangle\mathbb{1})(\hat{A} - \langle\hat{A}\rangle\mathbb{1})$$
$$\hat{A}\hat{B} - \langle\hat{A}\rangle\hat{B} - \hat{A}\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle\mathbb{1} - \hat{B}\hat{A} + \langle\hat{B}\rangle\hat{A} + \hat{B}\langle\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\mathbb{1}$$

$$[\Delta\hat{A}, \Delta\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}] \Rightarrow$$

$$|\langle\Delta A\,\Delta B\rangle|^2 = \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2$$ → lo descarto y hago más fuerte la desigualdad

$$\boxed{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2}$$

c)

① $\Delta A|\alpha\rangle = \lambda\,\Delta B|\alpha\rangle \qquad \lambda \in \mathfrak{Im} \Rightarrow \lambda = i|\lambda|$, $|\lambda| \in \mathbb{R}$

$\hat{A}, \hat{B}$ observables ⇒ son hermíticos, $\langle\hat{A}\rangle \in \mathbb{R}$, $\langle\hat{B}\rangle \in \mathbb{R}$

$$\langle\alpha|(\Delta A)^2|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle \geq |\langle\alpha|\Delta A\,\Delta B|\alpha\rangle|^2$$

$\langle\alpha|\Delta A\,\Delta A|\alpha\rangle$

Usamos → $\langle\alpha|\Delta A = \lambda^*\langle\alpha|\Delta B$ →

$$\lambda^*\langle\alpha|\Delta B\,\lambda\,\Delta B|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle \geq |\lambda^*\langle\alpha|(\Delta B)^2|\alpha\rangle|^2$$

$$|\lambda|^2\langle\alpha|(\Delta B)^2|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle \geq |\lambda|^2|\langle\alpha|(\Delta B)^2|\alpha\rangle|^2$$

$$|\lambda|^2|\langle\alpha|(\Delta B)^2|\alpha\rangle|^2 \geq |\lambda|^2|\langle\alpha|(\Delta B)^2|\alpha\rangle|^2$$

⇒ **Vale la igualdad** si se da ①

d) $\rightsquigarrow \langle x'|\alpha\rangle = (2\pi d^2)^{-\frac{1}{4}} \cdot e^{\left(i\frac{\langle p\rangle x'}{\hbar} - \frac{[x'-\langle x\rangle]^2}{4d^2}\right)}$

Función de onda $\psi_\alpha(x')$ para el estado $|\alpha\rangle$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4} \qquad \langle x'|\Delta x|\alpha\rangle = c\,\langle x'|\Delta p|\alpha\rangle$$

$c \in \Im m$

$$\Delta x = \hat{x} - \langle\hat{x}\rangle\mathbb{1} \qquad \langle\alpha|\Delta x\,\Delta x|\alpha\rangle = \langle\alpha|(\hat{x}-\langle\hat{x}\rangle\mathbb{1})(\hat{x}-\langle\hat{x}\rangle\mathbb{1})|\alpha\rangle$$

$$\Delta p = \hat{p} - \langle\hat{p}\rangle\mathbb{1}$$

$$\left(\langle\alpha|x - \langle\alpha|\langle\hat{x}\rangle\right)\left(x|\alpha\rangle - \langle\hat{x}\rangle|\alpha\rangle\right)$$

$$(\Delta x)^2 = \hat{x}^2 - 2\langle\hat{x}\rangle\hat{x} + \langle\hat{x}\rangle^2\mathbb{1} \rightarrow$$

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

$$\langle(\Delta x)^2\rangle \equiv \langle\alpha|(\Delta x)^2|\alpha\rangle \quad \Rightarrow$$

$$\langle(\Delta x)^2\rangle = \int\int dx'\,dx''\,\langle\alpha|x'\rangle\langle x'|(\Delta x)^2|x''\rangle\langle x''|\alpha\rangle$$

$$= \int\int dx'\,dx''\,(2\pi d^2)^{-\frac{1}{4}}\cdot e^{-i\frac{\langle p\rangle x'}{\hbar}}\cdot e^{-\frac{(x'-\langle x\rangle)^2}{4d^2}}\langle x'|(\Delta x)^2|x''\rangle\,(2\pi d^2)^{-\frac{1}{4}}\cdot e^{i\frac{\langle p\rangle x''}{\hbar}}\cdot e^{-\frac{(x''-\langle x\rangle)^2}{4d^2}}$$

$$= \int\int dx'\,dx''\,(2\pi d^2)^{-\frac{1}{2}}\,e^{i\frac{\langle p\rangle}{\hbar}(x''-x')}\cdot e^{-\frac{1}{4d^2}\left([x''-\langle x\rangle]^2+[x'-\langle x\rangle]^2\right)}\langle x'|(\Delta x)^2|x''\rangle$$

$$\langle x'|(\Delta x)^2|x''\rangle = \langle x'|x^2|x''\rangle - \langle x'|2\langle\hat{x}\rangle\,\hat{x}|x''\rangle + \langle x'|\langle\hat{x}\rangle^2\mathbb{1}|x''\rangle$$

$$\langle x'|x''|x''\rangle - 2\langle x\rangle x''\langle x'|x''\rangle + \langle\hat{x}\rangle^2\langle x'|x''\rangle$$

$$x''^2\langle x'|x''\rangle - 2\langle x\rangle x''\langle x'|x''\rangle + \langle\hat{x}\rangle^2\langle x'|x''\rangle$$

es un #

$$(x''^2 - 2\langle\hat{x}\rangle x'' + \langle\hat{x}\rangle^2)\underbrace{\langle x'|x''\rangle}_{\delta_{x'x''}} = (x''-\langle x\rangle)^2\,\delta_{x'x''}$$

$$(x''-\langle x\rangle)^2 \;\text{"="}\; (\langle x^2\rangle - \langle x\rangle^2)$$

$$\langle(\Delta x)^2\rangle = (2\pi d^2)^{-\frac{1}{2}}\int dx''\,\cdot e^{-\frac{2}{4d^2}[x''-\langle x\rangle]^2}(x''^2 - 2\langle x\rangle x'' + \langle x\rangle^2)$$

$$\langle(\Delta x)^2\rangle = \frac{1}{(2\pi d^2)^{1/2}}\int dx''\cdot e^{-\frac{1}{2d^2}(x''-\langle x\rangle)^2}[x''-\langle x\rangle]^2$$

$$\int_{-\infty}^{+\infty} du\,.\,e^{-\frac{u^2}{2d^2}}u^2 \qquad \beta = \frac{1}{2d^2}$$

$$\frac{d}{d\beta}\left(e^{-u^2\beta}\right) = -e^{-u^2\beta}\cdot u^2$$

NOTA

$$\int_{-\infty}^{+\infty} e^{-u^2\alpha}\,du = \sqrt{\frac{\pi}{\alpha}}$$

$$\frac{d}{d\beta}\left(-\int_{-\infty}^{+\infty} e^{-u^2\beta}\,du\right) =$$

$$-\frac{d}{d\beta}\left(\sqrt{\frac{\pi}{\beta}}\right) = -\sqrt{\pi}\cdot\left(-\frac{1}{2}\right)\beta^{-3/2} = \frac{\sqrt{\pi}}{(\sqrt{\beta})^3}\frac{1}{2}$$

$$\langle(\Delta x)^2\rangle = \frac{1}{\sqrt{2\pi d^2}}\,\frac{\sqrt{\pi}}{2}\,\frac{1}{\left(\sqrt{\frac{1}{2d^2}}\right)^3} = \frac{(\sqrt{2})^3\,d^3\sqrt{\pi}}{\sqrt{2\pi}\,d\,2} = d^2$$

$$p^2 - 2\langle p\rangle p + \langle p\rangle^2$$

$$\langle \alpha | (\Delta p)^2 | \alpha \rangle = \iint dp'\, dp'' \langle \alpha | p' \rangle \langle p' | (\Delta p)^2 | p'' \rangle \langle p'' | \alpha \rangle$$

análogamente será

$$(p''^2 - 2\langle p\rangle p'' + \langle p\rangle^2)\langle p' | p'' \rangle = (p'' - \langle p\rangle)^2 \, \delta_{p'p''}$$

$$= \iint dp'\, dp''\, dx'\, dx'' \, \langle \alpha | x' \rangle \langle x' | p' \rangle \langle p' | (\Delta p)^2 | p'' \rangle \langle p'' | x'' \rangle \langle x'' | \alpha \rangle$$

$$= \iint dp' dp'' dx' dx'' \frac{1}{(2\pi\hbar)^{1/4}} e^{-\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i p' x'}{\hbar}} \langle p' | (\Delta p)^2 | p'' \rangle \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i p'' x''}{\hbar}} \frac{1}{(2\pi\hbar)^{1/4}} e^{\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}}$$

$$\frac{1}{(2\pi\hbar)^{3/2}} \iiiint dp' dp'' dx' dx'' \, e^{-\frac{i\langle p\rangle x'}{\hbar} - \frac{i p' x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}} \langle p' | (\Delta p)^2 | p'' \rangle \, e^{\frac{i p'' x''}{\hbar} + \frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}}$$

$$\langle p' | p^2 - 2\langle p\rangle p + \langle p\rangle \mathbb{1} | p'' \rangle = p''^2 \langle p' | p'' \rangle - 2\langle p\rangle p'' \langle p' | p'' \rangle + \langle p\rangle \langle p' | p'' \rangle$$

$$= (p''^2 - 2\langle p\rangle p'' + \langle p\rangle)\, \delta(p' - p'')$$

$$\frac{i\langle p\rangle x'}{} - \frac{i p' x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2} \quad e^{\frac{i p' x''}{\hbar} + \frac{i\langle p\rangle x''}{\hbar} - (x''}$$

$$\langle \alpha | p^2 | \alpha \rangle = \iint dp'\, dp'' \langle \alpha | p' \rangle \langle p' | p^2 | p'' \rangle \langle p'' | \alpha \rangle$$

$$\iint dp' dp'' \langle \alpha | p' \rangle \, p''^2 \, \delta(p'' - p') \langle p'' | \alpha \rangle$$

$$\iiiint dx' dx'' dp' dp'' \langle \alpha | x' \rangle \langle x' | p' \rangle p''^2 \delta(p'' - p') \langle p'' | x'' \rangle \langle x'' | \alpha \rangle$$

$$\iiint dx' dx'' dp' \; p'^2 \; \langle \alpha | x' \rangle \langle x'' | \alpha \rangle \langle x' | p' \rangle \langle p' | x'' \rangle$$

$$\iint dx' dx'' \langle \alpha | x' \rangle \langle x'' | \alpha \rangle \int dp' \, p'^2 \frac{e^{\frac{i p'}{\hbar}(x' - x'')}}{(2\pi\hbar)}$$

$$\int dp\, p^2 e^{p\Delta x} = \frac{\partial^2}{\partial(\Delta x)^2}\left(\int dp \, e^{p\Delta x}\right) = \frac{\partial^2}{\partial(\Delta x)^2}\left(\int dp \; e^{i/\hbar \, p(x'-x'')}\right) \quad \Bigg\| \quad \frac{\partial^2 (e^{p\Delta x})}{\partial(\Delta x)^2} = p^2 \cdot e^{p\Delta x}$$

$$\frac{\partial^2}{\partial(\Delta x)^2} 2\pi\hbar \, \delta(\underbrace{x' - x''}_{\Delta x \cdot \frac{\hbar}{i}}) = -i\hbar^2 2\pi \frac{\partial^2}{\partial \xi^2} \delta(\xi)$$

$$\langle\alpha|(\Delta p)^2|\alpha\rangle = \langle\alpha|\, p^2 - 2p\langle p\rangle + \langle p\rangle^2 \,|\alpha\rangle$$

$$= \langle\alpha|\hat{p}^2|\alpha\rangle - 2\langle p\rangle\langle\alpha|\hat{p}|\alpha\rangle + \langle p\rangle^2\langle\alpha|\alpha\rangle$$

$$= \int dx' \langle\alpha|\hat{p}^2|x'\rangle\langle x'|\alpha\rangle - 2\langle p\rangle\int dx' \langle\alpha|\hat{p}|x'\rangle\langle x'|\alpha\rangle + \langle p\rangle^2$$

$$= \int dx' \left(\hbar^2 \frac{\partial^2}{\partial x'^2}\langle\alpha|x'\rangle\right)\langle x'|\alpha\rangle - 2\langle p\rangle\int dx' \left(i\hbar\frac{\partial}{\partial x'}\langle\alpha|x'\rangle\right)\langle x'|\alpha\rangle + \langle p\rangle^2$$

$$\langle x'|\alpha\rangle = \frac{1}{(2\pi d^2)^{1/4}} e^{\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}}$$

$$\langle x'|\alpha\rangle = \frac{1}{(2\pi d^2)^{1/4}}\, e^{\frac{i\langle p\rangle x'}{\hbar}}\left(\frac{i\langle p\rangle}{\hbar}\right) e^{-\frac{(x'-\langle x\rangle)^2}{4d^2}} + e^{\frac{i\langle p\rangle x'}{\hbar}} \cdot e^{-\frac{(x'-\langle x\rangle)^2}{4d^2}} \cdot \frac{2(x'-\langle x\rangle)}{4d^2}$$

$$\frac{1}{(2\pi d^2)^{1/4}} \cdot e^{\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}} \left[\frac{i\langle p\rangle}{\hbar} - \frac{\overset{1}{\cancel{2}}(x'-\langle x\rangle)}{\underset{2}{\cancel{4}}d^2}\right]$$

$$\frac{\partial}{\partial x'}\langle x'|\alpha\rangle = \langle x'|\alpha\rangle\left(\frac{i\langle p\rangle}{\hbar} - \frac{(x'-\langle x\rangle)}{2d^2}\right)$$

$$\frac{\partial^2}{\partial x'}\langle x'|\alpha\rangle = \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\left(\frac{i\langle p\rangle}{\hbar} - \frac{(x'-\langle x\rangle)}{2d^2}\right) + \langle x'|\alpha\rangle\left(-\frac{1}{2d^2}\right)$$

$$= \langle x'|\alpha\rangle\left(\frac{i\langle p\rangle}{\hbar} - \frac{(x'-\langle x\rangle)}{2d^2}\right)^2 - \frac{1}{2d^2}\langle x'|\alpha\rangle$$

$$= \langle x'|\alpha\rangle\left[\left(\frac{i\langle p\rangle}{\hbar} - \frac{[x'-\langle x\rangle]}{2d^2}\right)^2 - \frac{1}{2d^2}\right]$$

16.

a)

$$\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

estado $|S_z;+\rangle$

$$|S_z;+\rangle = |+\rangle$$

$$\langle S_x \rangle = \langle S_z;+|S_x|S_z;+\rangle = \langle +|S_x|+\rangle = \langle +|\frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)|+\rangle$$

$$\langle S_x \rangle = \frac{\hbar}{2}(\langle +|+\rangle\langle -|)|+\rangle = 0 \quad \rightarrow \quad \langle S_x \rangle^2 = 0$$

$$S_x . S_x = \frac{\hbar^2}{4}(|+\rangle\langle -| + |-\rangle\langle +|)(|+\rangle\langle -| + |-\rangle\langle +|)$$

$$= \frac{\hbar^2}{4}(|-\rangle\langle +|+\rangle\langle -| + |+\rangle\langle -|-\rangle\langle +|) = \frac{\hbar^2}{4}(|-\rangle\langle -| + |+\rangle\langle +|)$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4}\langle +|(|-\rangle\langle -| + |+\rangle\langle +|)|+\rangle = \frac{\hbar^2}{4}$$

$$\Rightarrow \quad \boxed{\langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4}}$$

$$\langle (\Delta S_y)^2 \rangle = \langle S_y^2 \rangle - \langle S_y \rangle^2$$

$$\langle S_y \rangle = \langle +|S_y|+\rangle = \langle +|\frac{i\hbar}{2}|-\rangle = 0$$

$$S_y . S_y = \left(\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{2}\right)\cdot(-|+\rangle\langle -| + |-\rangle\langle +|)(-|+\rangle\langle -| + |-\rangle\langle +|)$$

$$= \frac{i^2\hbar^2}{4}\cdot(-|-\rangle\langle -| - |+\rangle\langle +|) = \frac{\hbar^2}{4}(|-\rangle\langle -| + |+\rangle\langle +|)$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4}\langle +|(|-\rangle\langle -| + |+\rangle\langle +|)|+\rangle = \frac{\hbar^2}{4}$$

$$\boxed{\langle (\Delta S_y)^2 \rangle = \frac{\hbar^2}{4}}$$

$$\therefore \quad \langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \frac{1}{4}|\langle [S_x, S_y] \rangle|^2$$

$$\frac{\hbar^2}{4}\cdot\frac{\hbar^2}{4} \geq \frac{1}{4}|\langle i\hbar S_z \rangle_{|+\rangle}|^2 = \frac{\hbar^2}{4}\left|\frac{\hbar}{2}\right|^2 = \frac{\hbar^4}{16}$$

$$\langle S_z \rangle_{|+\rangle} = \langle +|S_z|+\rangle$$

Se verifica la igualdad en la relación de incerteza mínima

b)

$$|S_x;+\rangle = \frac{|+\rangle}{\sqrt{2}} + \frac{|-\rangle}{\sqrt{2}}$$

$$\langle S_x \rangle = \frac{\langle +| + \langle -|}{\sqrt{2}}\left[\frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)\right]\frac{|-\rangle + |+\rangle}{\sqrt{2}}$$

$|+\rangle \; |-\rangle$

$$\langle S_x \rangle = \frac{\hbar}{4}\left(\langle -| + \langle +|\right)\left(|+\rangle + |-\rangle\right) = \frac{\hbar}{2}$$

$$\langle S_x^2 \rangle = \frac{\langle -| + \langle +|}{\sqrt{2}}\left(\frac{\hbar^2}{4}\left[|+\rangle\langle +| + |-\rangle\langle -|\right]\right)\frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$= \frac{1}{2}\cdot\frac{\hbar^2}{4}\left(\langle +| + \langle -|\right)\left(|+\rangle + |-\rangle\right) = \frac{1}{2}\cdot\frac{\hbar^2}{4}\cdot 2$$

$$\langle \Delta S_x \rangle = \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0$$

$$\langle S_y \rangle = \frac{1}{2}\left(\langle +| + \langle -|\right)\left(\frac{i\hbar}{2}\left[-|+\rangle\langle -| + |-\rangle\langle +|\right]\right)\left(|-\rangle + |+\rangle\right)$$

$$= \frac{1}{2}\cdot\frac{i\hbar}{2}\left(-\langle -| + \langle +|\right)\left(|-\rangle + |+\rangle\right) = \frac{i\hbar}{4}(-1+1) = 0$$

$$\langle S_y^2 \rangle = \frac{1}{2}\left(\langle +| + \langle -|\right)\left(\frac{\hbar^2}{4}\left(|-\rangle\langle -| + |+\rangle\langle +|\right)\right)\left(|+\rangle + |-\rangle\right)$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{8}\left(\langle -| + \langle +|\right)\left(|+\rangle + |-\rangle\right) = \frac{\hbar^2}{4}$$

$$\boxed{\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = 0}$$

$\neq 0$

respecto al autoestado $|S_x = \hbar/2\rangle$

pues $[S_x, S_y] = i\hbar S_z$ y $S_z |S_x = \hbar/2\rangle = 0$

Remark – S_x, S_y conmutarán respecto a un autoestado de S_x pues la $\langle(\Delta S_x)^2\rangle = 0$ en dicha situación

17.

CL de $|+\rangle$ y $|-\rangle$ que hace máximo

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle$$

$$|\alpha\rangle = a|+\rangle + b|-\rangle \quad \text{con} \quad a^2 + b^2 = 1$$

(normalización)

$$\langle \alpha | (\Delta S_x)^2 | \alpha \rangle \langle \alpha | (\Delta S_y)^2 | \alpha \rangle \geq \frac{1}{4} |\langle [S_x, S_y] \rangle|^2$$

$2a\,da + 2b\,db = 0$

$a\,da = -b\,db$ ← vínculo

$$\Rightarrow \quad |\langle i\hbar S_z \rangle|^2 = |i\hbar|^2 |\langle S_z \rangle_\alpha|^2$$

$$\hbar^2 \left| \frac{\hbar}{2}\left(\left(a^*\langle +| + b^*\langle -|\right)\left(|+\rangle\langle +| - |-\rangle\langle -|\right)\left(a|+\rangle + b|-\rangle\right)\right) \right|^2$$

$$\frac{\hbar^4}{4} \left| \left(a^*\langle +| + b^*\langle -|\right)\left(a|+\rangle - b|-\rangle\right) \right|^2$$

$$\frac{\hbar^4}{4} \left| \left(a^* a - b^* b\right) \right|^2$$

$$|\langle [S_x, S_y] \rangle|^2 = \frac{\hbar^4}{4} \left| |a|^2 - |b|^2 \right|^2$$

Esto es máximo para $|b| = 0 \rightarrow$

$|a| = 1 \rightarrow$

en general: $a = \cos\phi + i\,\text{sen}\,\phi$ con $a \in \mathbb{C}$

$$\boxed{\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle_{máx} = \frac{\hbar^4}{16}}$$

con $\begin{cases} |\alpha\rangle = |+\rangle \\ |\alpha\rangle = |-\rangle \end{cases}$

Este valor es justamente el valor de incerteza que se obtiene para el mínimo $\Rightarrow$ No se viola la relación de incerteza

18.

$\langle b'|A|b''\rangle$ — base $\{|b'\rangle\}$ (No autoestados)

Pero puedo pasar a una base $\{|c'\rangle\}$ con un operador U unitario:

$|c'\rangle = \hat{U}|b'\rangle$

$|c^j\rangle = \sum_l^N |c^l\rangle\langle b^l|b^j\rangle$

$\langle c^j| = \sum_l^N (\langle c^l|)(\langle b^l|b^j\rangle)^*$

$\sum(\langle c^l|)(\langle b^j|b^l\rangle)$

$\langle c^j| = \sum_l^N \langle b^j|b^l\rangle\langle c^l|$

→ traspaso a la nueva base $\{|c'\rangle\}$

$\langle b'|A|b''\rangle$ es real ⇒ (elementos de la matriz son $\mathbb{R}$)

$\langle b'|A|b''\rangle = (\langle b'|A|b''\rangle)^*$

$\langle b'|A|b''\rangle = \langle b''|A^\dagger|b'\rangle$

$\langle b^j|A|b^k\rangle = \langle b^k|A^\dagger|b^j\rangle$

Sea el operador en la base $\{|c'\rangle\}$

$\langle c^k|A|c^j\rangle \Rightarrow$

$(\langle c^k|A|c^j\rangle)^* = \langle c^j|A^\dagger|c^k\rangle =$

$\sum_{m,l} \underbrace{\langle c^j|b^m\rangle}_{\#}(\langle b^m|)(A^\dagger|b^l\rangle)\underbrace{\langle b^l|c^k\rangle}_{\#}$ ← Aplico DC para sacar la daga de A

$\sum_{m,l} \langle c^k|b^l\rangle\langle b^l|A|b^m\rangle\langle b^m|c^j\rangle$

$\langle c^k|A|c^j\rangle = (\langle c^k|A|c^j\rangle)^* \Rightarrow$ **los elementos de la matriz son reales**

Aún cuando usamos otra base permanecen reales

- Verificación con S_y, S_z

base $= \{\underbrace{a|+\rangle}_{c^1}, \underbrace{b|-\rangle}_{c^2}\}$ → base general — a, b son # cumpliendo que: $|a|^2+|b|^2 = 1$

$\langle c^l|S_z|c^k\rangle \rightarrow$

$$S_{z\,(\text{base } c)} \doteq \begin{pmatrix} |a|^2\langle +|S_z|+\rangle & a.b\langle +|S_z|-\rangle \\ b.a\langle -|S_z|+\rangle & |b|^2\langle -|S_z|-\rangle \end{pmatrix}$$

$$S_{z\,(\text{base } c)} \doteq \frac{\hbar}{2}\begin{pmatrix} |a|^2 & 0 \\ 0 & -|b|^2 \end{pmatrix}$$

⇒ todos los elementos son reales para cualquier elección de a, b ⇒ S_z es $\mathbb{R}$

$\langle c^l|S_y|c^k\rangle$

$$S_{y\,(\text{base } c)} \doteq \begin{pmatrix} |a|^2\langle +|S_y|+\rangle & a.b\langle +|S_y|-\rangle \\ b.a\langle -|S_y|+\rangle & |b|^2\langle -|S_z|-\rangle \end{pmatrix}$$

$$S_{y\,(\text{base } c)} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -a.b.i \\ b.a.i & 0 \end{pmatrix}$$

⇒ Los elementos no son $\mathbb{R}$ para ciertas elecciones de (a,b) ⇒ $S_y \notin \mathbb{R}$

19. La base donde un operador es diagonal es la base que lo diagonaliza, es decir la base de autoestados de dicho operador

$$S_z \underset{\{|c\rangle\}}{\doteq} \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \{|c\rangle\} = \{|+\rangle, |-\rangle\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

$$S_x \underset{\{|c'\rangle\}}{\doteq} \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \{|c'\rangle\} = \left\{\frac{|+\rangle + |-\rangle}{\sqrt{2}}, \frac{|+\rangle - |-\rangle}{\sqrt{2}}\right\} = \left\{\begin{pmatrix}1\\1\end{pmatrix}; \begin{pmatrix}1\\-1\end{pmatrix}\right\}\frac{1}{\sqrt{2}}$$

autovectores normalizados

$$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}| & |\\ c_1 & c_2\\ | & |\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}| & |\\ c_1 & c_2\\ | & |\end{pmatrix}$$

autovectores

- <u>Relación General</u>

$$U = \sum_{\ell}^{2} |c'^{\ell}\rangle\langle c^{\ell}| \rightarrow |c'^{\delta}\rangle = \sum_{\ell}^{2} |c^{\ell}\rangle\langle c^{\ell}|c'^{\delta}\rangle$$

(nueva: $|c'^{\delta}\rangle$; vieja: $|c^{\ell}\rangle$)

$$\langle c^{k}|c'^{\delta}\rangle = \langle c^{k}|U|c^{\delta}\rangle \quad \text{transformación}$$

(base vieja)

$$\underset{\{|c\rangle\}\rightarrow\{|c'\rangle\}}{U} = \begin{pmatrix} \left(\frac{\langle +| + \langle -|}{\sqrt{2}}\right)|+\rangle & \left(\frac{\langle +| + \langle -|}{\sqrt{2}}\right)|-\rangle \\ \left(\frac{\langle +| - \langle -|}{\sqrt{2}}\right)|+\rangle & \left(\frac{\langle +| - \langle -|}{\sqrt{2}}\right)|-\rangle \end{pmatrix} = \begin{pmatrix} \frac{\langle +|+\rangle}{\sqrt{2}} & + \frac{\langle -|-\rangle}{\sqrt{2}} \\ \frac{\langle +|+\rangle}{\sqrt{2}} & - \frac{\langle -|-\rangle}{\sqrt{2}} \end{pmatrix}$$

$$\boxed{U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}$$

- <u>Construcción a Manopla</u>

Usamos $|+\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |-\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right) = \left(\begin{bmatrix}\frac{U_{11}}{\sqrt{2}} + \frac{U_{12}}{\sqrt{2}}\\ \frac{U_{21}}{\sqrt{2}} + \frac{U_{22}}{\sqrt{2}}\end{bmatrix} \begin{bmatrix}\frac{U_{11}}{\sqrt{2}} - \frac{U_{12}}{\sqrt{2}}\\ \frac{U_{21}}{\sqrt{2}} - \frac{U_{22}}{\sqrt{2}}\end{bmatrix}\right)$$

$$U_{11} + U_{12} = \sqrt{2} \qquad U_{11} = U_{12}$$
$$U_{21} = -U_{22} \qquad U_{21} - U_{22} = \sqrt{2}$$

$$\rightarrow \quad U_{11} = \frac{\sqrt{2}}{2} = U_{12} \qquad -U_{22} = \frac{\sqrt{2}}{2} = U_{21}$$

$$U = \begin{pmatrix}\sqrt{2}/2 & \sqrt{2}/2\\ \sqrt{2}/2 & -\sqrt{2}/2\end{pmatrix}$$

$$\underset{\substack{\text{en base}\\ |+\rangle\\ |-\rangle}}{U} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$$

$$\boxed{U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}$$

$\Rightarrow$ El resultado es consistente

20. a) $\hat{A}$ es hermítico y $\hat{A}|a'\rangle = a'|a'\rangle$

Hay que evaluar → $\langle b''|f(A)|b'\rangle$

Conocemos U tal que:

$|b'\rangle = U|a'\rangle$

$U^{-1}|b'\rangle = |a'\rangle$

$|a'\rangle = U^{-1}|b'\rangle$

Matrices de transformación

Se puede poner como serie

Usamos que f es analítica

$$f(\hat{A}) = \sum_{i=0}^{\infty} c_i \hat{A}^i$$

$$U = \sum_l |b^l\rangle\langle a^l|$$

$|b'\rangle = \sum |b^l\rangle\langle a^l|a'\rangle$

$\langle b'| = \sum (\langle b^l|)(\langle a'|a^l\rangle)$

$\langle b'| = \sum \langle a'|a^l\rangle\langle b^l|$

$A|a'\rangle = a'|a'\rangle$

$A(A|a'\rangle) = A a'|a'\rangle = (a')^2|a'\rangle$

$A^n|a'\rangle = a'^n|a'\rangle \Rightarrow A^i|a'\rangle = a'^i|a'\rangle$

La idea es meter un $\mathbb{1}$ para poder operar

$$\langle b''|f(A)|b'\rangle = \sum_{i=0}^{\infty} \langle b^k|c_i A^i|b^j\rangle$$

$$= \sum_{i=0}^{\infty}\sum_{l}^{N} c_i \langle b^k|A^i|a^l\rangle\langle a^l|b^j\rangle$$

$$\sum_{i=0}^{\infty}\sum_{l}^{N} c_i \langle b^k|(a^l)^i|a^l\rangle\langle a^l|b^j\rangle$$

$$\sum_{i=0}^{\infty}\sum_{l}^{N} c_i (a^l)^i \langle b^k|a^l\rangle\langle a^l|b^j\rangle$$

$$\boxed{\langle b^k|f(A)|b^j\rangle = \sum_{l}^{N} f(a^l)\langle b^k|a^l\rangle\langle a^l|b^j\rangle}$$

Estos productos los conozco

$$|b^j\rangle = \sum_l^N |b^l\rangle\langle a^l|a^j\rangle$$

$\langle a^l|b^j\rangle = U|a^j\rangle$

$\langle a^l|b^j\rangle = \langle a^l|U|a^j\rangle$ → lo conozco

⇒ conozco todos los productillos $\langle a^l|b^j\rangle$

b)

$$\langle \vec{p}''|F(\hat{r})|\vec{p}'\rangle \qquad \hat{r} = \sqrt{x^2+y^2+z^2}$$

$\hat{x}, \hat{y}, \hat{z}$ operadores

$\langle p_x'', p_y'', p_z''|F(r)|p_x', p_y', p_z'\rangle$

$$F(\hat{r}) = \sum_{i=0}^{\infty} c_i (\sqrt{x^2+y^2+z^2})^i$$

$$= \int d^3x''' \langle \vec{p}''|F(\hat{r})|\vec{x}'''\rangle\langle \vec{x}'''|\vec{p}'\rangle$$

x

21.

a)

$$\{x, F(p_x)\}_{\text{clásico}} = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p_x} - \underbrace{\frac{\partial x}{\partial p_x}}_{=0}\frac{\partial F}{\partial x}$$

$$\boxed{\{x, F(p_x)\}_{\text{clásico}} = \frac{\partial F(p_x)}{\partial p_x}}$$

b)

$$\left[x, e^{\left(\frac{i p_x a}{\hbar}\right)}\right] = \left[x, \sum_{n=0}^{\infty} \frac{i^n p_x^n a^n}{n!}\frac{1}{\hbar^n}\right]$$

$$= \left[x, \sum_{n=0}^{\infty} \left(\frac{i p_x a}{\hbar}\right)^n \frac{1}{n!}\right]$$

$$= [x, 1] + \left[x, \frac{i p_x a}{\hbar}\right] + \left[x, \frac{i^2 p_x^2 a^2}{\hbar^2}\cdot\frac{1}{2}\right] + \left[x, \frac{i^3 p_x^3 a^3}{\hbar^3}\cdot\frac{1}{6}\right] + \dots + \left[x, \frac{i^n p_x^n a^n}{\hbar^n}\frac{1}{n!}\right]$$

Término general $\rightsquigarrow \frac{i^n a^n}{\hbar^n n!}[x, p_x^n]$

$$[x, p_x] = i\hbar$$

$$[x, p_x^2] = [x, p_x \cdot p_x] = p_x[x, p_x] + [x, p_x]p_x = -i\hbar\frac{\partial}{\partial x}(i\hbar) + (i\hbar)\left(-i\hbar\frac{\partial}{\partial x}\right) = \hbar^2\frac{\partial}{\partial x} + \hbar^2\frac{\partial}{\partial x} = 2\hbar^2\frac{\partial}{\partial x}$$

$$[x, p_x^3] = [x, \underbrace{p_x \cdot p_x}_{p_x^2} \cdot p_x] \overset{?}{=} p_x^2([x, p_x]) + ([x, p_x^2])p_x = -\hbar^2\frac{\partial^2}{\partial x^2}(i\hbar) + 2\hbar^2\frac{\partial}{\partial x}(i\hbar)\frac{\partial}{\partial x}$$

$$= -i3\hbar^3\frac{\partial^2}{\partial x^2}$$

$$[x, p_x^4] = [x, p_x^3 \cdot p_x] =$$

$$= p_x^3([x, p_x]) + ([x, p_x^3])p_x =$$

$$= i\hbar^3\frac{\partial^3}{\partial x^3}(i\hbar) + \left(-i3\hbar^3\frac{\partial}{\partial x^2}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)$$

$$= -\hbar^4\frac{\partial^3}{\partial x^3} + -3\hbar^4 1\frac{\partial^3}{\partial x^3} = -4\hbar^4\cdot\frac{\partial^3}{\partial x^3}$$

nota

$$\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right) = -\hbar^2\frac{\partial^2}{\partial x^2}$$

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^3 = +i\hbar^3\frac{\partial^3}{\partial x^3}$$

$$(-i)^3 = (-)^3 i^3 = -i^2 i = i$$

$$i2\hbar \cdot p_x^1$$

$$[x,p_x^2] = 2\hbar^2 \frac{\partial}{\partial x} = \frac{2\hbar}{-i}\left(-i\hbar\frac{\partial}{\partial x}\right) \Rightarrow [x,p_x] = i\hbar = i\,1\,\hbar^1 \frac{\partial^0}{\partial x^0} = i\hbar\, p_x^0$$

$$[x,p_x^3] = -i3\hbar^3 \frac{\partial^2}{\partial x^2} = 3i\hbar\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) \to i\hbar 3 \cdot p_x^2$$

$$[x,p_x^4] = -4\hbar^4 \frac{\partial^3}{\partial x^3} = -\frac{4\hbar}{i}\left(i\hbar^3\frac{\partial^3}{\partial x^3}\right) \to i4\hbar \cdot p_x^3$$

$$[x,p_x^n] = i\hbar\, n\, p_x^{n-1}$$

$$[x, e^{\frac{ip_x a}{\hbar}}] = \sum_{n=0}^{\infty} \frac{i^n a^n}{\hbar^n\, n!} \cdot i\hbar \underbrace{n p_x^{n-1}}_{\frac{\partial p_x^n}{\partial p_x}} = \sum_{n=0}^{\infty}\left(\frac{ia}{\hbar}\right)^n \frac{1}{n!} \cdot i\hbar \frac{\partial p_x^n}{\partial p_x}$$

- Comparación

Sea ahora $F(p_x) = e^{\frac{ip_x a}{\hbar}}$

$$\frac{\partial F}{\partial p_x} = e^{\frac{ip_x a}{\hbar}} \cdot \frac{ia}{\hbar}$$

$$[x, e^{\frac{ip_x a}{\hbar}}] = \sum \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n i\hbar \frac{\partial p_x^n}{\partial p_x}$$

$$\frac{[\quad]}{i\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n \frac{\partial p_x^n}{\partial p_x}$$

$$\frac{\partial}{\partial p_x}\left(\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{ia}{\hbar}p_x\right)^n\right)$$

$$\frac{\partial}{\partial p_x}\left(e^{\frac{iap_x}{\hbar}}\right)$$

$$\{x, e^{\frac{ip_x a}{\hbar}}\}_{\text{clásico}} = \frac{ia}{\hbar} \cdot e^{\frac{ip_x a}{\hbar}}$$

$$[\quad] = \{\quad\}_{\text{clásico}} \cdot i\hbar$$ (conmutador)

$$[x, e^{\frac{ip_x a}{\hbar}}] = -a \cdot e^{\frac{ip_x a}{\hbar}}$$

Coinciden perfectamente, comprobándose la presunción de Dirac

$$\boxed{\{\quad\}_{\text{clásico}} = \frac{[\quad]}{i\hbar} = e^{\frac{iap_x}{\hbar}} \cdot \frac{ia}{\hbar}}$$

c) Probar que $e^{\left(\frac{ip_x a}{\hbar}\right)}|x'\rangle$ es autoestado del operador $\hat{x}$ con $\hat{x}|x'\rangle = x'|x'\rangle$

$$e^{\frac{ip_x a}{\hbar}} = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\left(\frac{ia}{\hbar}\right)^n p_x^n \to$$

$$\frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n p_x^n |x'\rangle = \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n -i\hbar\left(\frac{\partial^n}{\partial x^n}\right)|x'\rangle$$

→ no sabemos como operar esto

$$-\frac{[\hat{x}, e^{i\hat{p}_x\frac{a}{\hbar}}]}{a}|x'\rangle =$$

$$-\frac{\hat{x}\cdot e^{i\hat{p}_x\frac{a}{\hbar}}}{a}|x'\rangle + \frac{e^{i\hat{p}_x\frac{a}{\hbar}}}{a}\overbrace{\hat{x}|x'\rangle}^{x'|x'\rangle} = e^{i\hat{p}_x\frac{a}{\hbar}}|x'\rangle$$

$$\hat{x}\cdot\underbrace{\left(e^{i\hat{p}_x\frac{a}{\hbar}}|x'\rangle\right)}_{|||} = x' e^{i\hat{p}_x\cdot\frac{a}{\hbar}}|x'\rangle - a\cdot e^{i\hat{p}_x\frac{a}{\hbar}}|x'\rangle$$

$$\hat{x}|\alpha\rangle = (x'-a)|\alpha\rangle \Rightarrow \boxed{\text{autovalor} = (x'-a)}$$

22. a)

$$[x_i, G(\vec{p})] = i\hbar \frac{\partial G}{\partial p_i} \qquad [p_i, F(\vec{x})] = -i\hbar \frac{\partial F}{\partial x_i}$$

F, G pueden expresarse en series de potencias de su argumento ⇒

$$G(\vec{p}) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\partial^n G}{\partial p_x^n} \cdot p_x^n + \frac{\partial^n G}{\partial p_y^n} p_y^n + \frac{\partial^n G}{\partial p_z^n} p_z^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n G}{\partial p_j^n} p_j^n$$

$$[x_i, \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n G}{\partial p_j^n} p_j^n] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n G}{\partial p_j^n} [x_i, p_j^n] = \begin{cases} 0 & si\ i \neq j \\ \neq 0 & si\ i = j \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n G}{\partial p_i} i\hbar\, n\, p_i^{n-1} \quad \left(n\, p_i^{n-1} \to \frac{\partial p_i^n}{\partial p_i}\right)$$

$$[x_i, G(\vec{p})] = i\hbar \frac{\partial}{\partial p_i} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n G}{\partial p_i} p_i^n$$

$$\boxed{[x_i, G(\vec{p})] = i\hbar \frac{\partial G}{\partial p_i}}$$

$$[p_i, F(\vec{x})] = [p_i, \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F}{\partial x_j^n} x_j^n] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F}{\partial x_j^n} [p_i, x_j^n]$$

$$[p_i, x_j^n] = \begin{cases} 0 & si\ i \neq j \\ \neq 0 & si\ i = j \end{cases} \rightarrow [p_i, x_i^n]\psi = -i\hbar \frac{\partial}{\partial x_i}(x_i^n) - x_i^n(-i\hbar)\frac{\partial}{\partial x_i}$$

$$= -i\hbar \left(n \cdot x_i^{n-1}\psi + x_i^n \cdot \frac{\partial \psi}{\partial x_i}\right) + i\hbar x_i^n \frac{\partial \psi}{\partial x_i}$$

$$= -i\hbar n \cdot x_i^{n-1}$$

$$[p_i, F(\vec{x})] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F}{\partial x_i^n} \left(-i\hbar\, n\, x_i^{n-1}\right) \quad \left(n\, x_i^{n-1} \to \frac{\partial x_i^n}{\partial x_i}\right)$$

$$-i\hbar \frac{\partial}{\partial x_i}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F}{\partial x_i^n} x_i^n\right) \Rightarrow$$

$$\boxed{[p_i, F(\vec{x})] = -i\hbar \frac{\partial F}{\partial x_i}}$$

b)

$$[x^2, p^2] = [x^2, p \cdot p] = p[x^2, p] + [x^2, p]p$$

$$= -p[p, x^2] - [p, x^2]p$$

$$= -p\left(x[p,x] + [p,x]x\right) - \left(x[p,x] + [p,x]x\right)p$$

$$= -px(-i\hbar) - px(-i\hbar) - x(-i\hbar)p - (-i\hbar)xp$$

$$= +i\hbar px + i\hbar px + i\hbar xp + i\hbar xp = i\hbar 2(p \cdot x + xp)$$

$$= 2i\hbar\left(-i\hbar\psi - i\hbar x \frac{\partial \psi}{\partial x} - i\hbar x \frac{\partial \psi}{\partial x}\right) = 2i\hbar\left(-i\hbar - 2i\hbar x \frac{\partial}{\partial x}\right)$$

$$(+2x)\left(-i\hbar \frac{\partial}{\partial x}\right)$$

$$= +2i\hbar\left(-i\hbar + 2xp\right)$$

$px(\psi) = x \frac{\partial \psi}{\partial x} + 1 \cdot \psi$ … xp

$(x \cdot p + 1)$

$[p,x] = px - xp$

$$[x^2,p^2] = +2i\hbar(-i\hbar + 2xp) = -2i^2\hbar^2 + 4i\hbar xp = 2\hbar^2 + i4\hbar xp$$

$$\{x^2,p^2\}_{CLÁSICO} = \frac{\partial x^2}{\partial x}\cdot\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\cdot\frac{\partial p^2}{\partial x} \qquad = 2\hbar^2 + 4\hbar^2 x\frac{\partial}{\partial x}$$

($\frac{\partial x^2}{\partial p} = 0$)

$$[x^2,p^2] = 4i\hbar xp + 2\hbar^2$$

$$= 2x \cdot 2p = 4xp \rightarrow$$

$$i\hbar\{x^2,p^2\} = i\hbar 4xp = i\hbar 2\left(2x \cdot -i\hbar\frac{\partial}{\partial x}\right) = -i^2 4x\hbar^2\frac{\partial}{\partial x}$$

$$i\hbar\{x^2,p^2\} = 4\hbar^2 x\frac{\partial}{\partial x}$$

$$i\hbar\{x^2,p^2\} = 4i\hbar xp$$

$[x,p] = x.p - p.x$
$\{x,p\} = x.p + p.x$

No coinciden aparentemente,

pero esto es un problema de pensar que en classical Mechanics xp conmuta y es lo mismo que px. Entonces podemos reescribir el $\{\ \}_{clásico}$ como:

$$2i\hbar(xp + px)$$

⇓

$$\frac{[\ ,\ ]}{i\hbar} = \{\ \}_{CLÁSICO}$$

⇓

Son iguales

* Otro modo inteligente de calcular $[x^2,p^2]$ con lo hecho en parte a)

$$[x^2,p^2] = -[p^2, x.x] = -[G(p), x.x'] = -\left(x[G(p),x] + [G(p),x]x\right)$$

$$= x[x,G(p)] + [G(p),x]x$$

$$= x\left(i\hbar\frac{\partial G}{\partial p}\right) + \left(i\hbar\frac{\partial G}{\partial p}\right).x$$

si $G = p^2$

$$= i\hbar(x.2p) + i\hbar 2p.x$$

$$= 2i\hbar(xp + px)$$

23.

$$\mathcal{I}(\vec{1}) = e^{\left(\frac{-i\hat{p}.\vec{1}}{\hbar}\right)}$$

donde $\hat{p}$ es operador impulso

→ Es una traslación finita

a) evaluar $[x_i, \mathcal{I}(\vec{1})]$

$$\mathcal{I}(\vec{1}) = \sum_{n=0}^{\infty}\left(\frac{-i\hat{p}.\vec{1}}{\hbar}\right)^n\cdot\frac{1}{n!}$$

$$\hat{p}.\Delta\vec{x} = p_x.\Delta x + p_y.\Delta y + p_z.\Delta z$$

$$\mathcal{I}(\vec{1}) = e^{-\frac{i}{\hbar}p_x\Delta x}.e^{-\frac{i}{\hbar}p_y\Delta y}.e^{-\frac{i}{\hbar}p_z\Delta z} = \left(\sum_{n=0}^{\infty}\left(\frac{-i\,p_x\Delta x}{\hbar}\right)^n\frac{1}{n!}\right)\left(\sum_{l=0}^{\infty}\left(\frac{-i\,p_y\Delta y}{\hbar}\right)^l\frac{1}{l!}\right)\cdot\left(\ \right)$$

$$\left(\sum_{m=0}^{\infty}\left(\frac{-i\,p_z\Delta z}{\hbar}\right)^m\frac{1}{m!}\right)$$

→ es un $\Delta\vec{x}$ traslación <u>finita</u>

Ahora cambiaremos de notación tomando:

$$\Delta\vec{x} = \Delta x\hat{x} + \Delta y\hat{y} + \Delta z\hat{z} = \Delta x_i\hat{\imath} + \Delta x_j\hat{\jmath} + \Delta x_k\hat{k} \Rightarrow$$

$$\mathcal{I}(\Delta\vec{x}) = \left[\sum_{n=0}^{\infty}\left(\frac{-i\,p_{x_i}\Delta x_i}{\hbar}\right)^n \cdot \frac{1}{n!}\right]\left[\sum_{l=0}^{\infty}\left(\frac{-i\,p_{x_j}\Delta x_j}{\hbar}\right)^l \frac{1}{l!}\right]\left[\sum_{m=0}^{\infty}\left(\frac{-i\,p_{x_k}\Delta x_k}{\hbar}\right)^m \frac{1}{m!}\right]$$

$$[x_i, \mathcal{I}(\Delta\vec{x})] = x_i \cdot \sum_n \sum_l \sum_m - \sum_n \sum_l \sum_m x_i$$
$$= \sum_l \sum_m \left(x_i \sum_n - \sum_n x_i\right)$$
$$= \sum_l \sum_m \left(\sum_n \left(\frac{-i\Delta x_i}{\hbar}\right)^n \frac{1}{n!}\, x_i p_{x_i}^n - \sum_n \left(\frac{-i\Delta x_i}{\hbar}\right)^n \cdot \frac{1}{n!}\, p_{x_i}^n x_i\right)$$
$$= \sum_l \sum_m \sum_n \left(\frac{-i\Delta x_i}{\hbar}\right)^n \cdot \frac{1}{n!}\,[x_i, p_{x_i}^n]$$

$$\downarrow$$
$$i\hbar\, n\, p_{x_i}^{n-1}$$
$$\downarrow$$

$$[x_i, \mathcal{I}(\Delta\vec{x})] = \sum_l \sum_m \sum_n \left(\frac{-i\Delta x_i}{\hbar}\right)^n \frac{1}{n!}\left(i\hbar \frac{\partial p_{x_i}^n}{\partial p_{x_i}}\right)$$

$$[x_i, \mathcal{I}(\Delta\vec{x})] = i\hbar \frac{\partial}{\partial p_{x_i}}\left(\sum_l \sum_m \sum_n\right) = i\hbar \frac{\partial}{\partial p_{x_i}}\left(e^{-i\frac{\vec{p}\cdot\Delta\vec{x}}{\hbar}}\right)$$

$$[x_i, \mathcal{I}(\Delta\vec{x})] = i\hbar\left(-\frac{i}{\hbar}\Delta x_i\right) e^{-i\frac{\hat{p}\cdot\Delta\vec{x}}{\hbar}}$$ (#)

$$\boxed{[x_i, \mathcal{I}(\Delta\vec{x})] = \Delta x_i \cdot \mathcal{I}(\Delta\vec{x})}$$

b)

¿Como $\langle\vec{x}\rangle$ cambia frente a traslaciones? $\mathcal{I}(\Delta\vec{x}) = e^{-i\frac{\hat{p}\cdot\Delta\vec{x}}{\hbar}}$

$\langle\vec{x}\rangle_\alpha = \langle\alpha|\hat{\vec{x}}|\alpha\rangle \rightarrow$ queremos ver $(\langle\alpha|\mathcal{I}^\dagger)\,\hat{x}\,(\mathcal{I}|\alpha\rangle)$, es decir $\langle\vec{x}\rangle_{\mathcal{I}(\Delta x)|\alpha\rangle}$

(*) sea 1D $\Rightarrow$

$$\langle\alpha|\, e^{\frac{i\hat{p}\cdot\Delta x}{\hbar}} \cdot \hat{x} \cdot e^{-i\frac{\hat{p}\cdot\Delta x}{\hbar}}\,|\alpha\rangle \Rightarrow$$ ($\uparrow \mathbb{1}$)

Ahora usamos la relación de conmutación hallada en parte a)

$$[\vec{x}, \mathcal{I}] = [x\hat{x} + y\hat{y} + z\hat{z}, \mathcal{I}] = [x\hat{x}, \mathcal{I}] + [y\hat{y}, \mathcal{I}] + [z\hat{z}, \mathcal{I}]$$
$$[\hat{\vec{x}}, \hat{\mathcal{I}}] = \Delta x\cdot\mathcal{I} + \Delta y\cdot\mathcal{I} + \Delta z\cdot\mathcal{I} = \hat{\vec{x}}\,\hat{\mathcal{I}}(\Delta\vec{x}) - \hat{\mathcal{I}}(\Delta\vec{x})\cdot\hat{\vec{x}} = \Delta\vec{x}\,\hat{\mathcal{I}}(\Delta\vec{x})$$
$$= \Delta\vec{x}\cdot\mathcal{I} = \vec{x}\cdot\mathcal{I} - \mathcal{I}\cdot\vec{x} \Rightarrow \quad \mathcal{I}\cdot\hat{\vec{x}} = \vec{x}\,\mathcal{I} - \Delta\vec{x}\,\mathcal{I}$$
$$\mathcal{I}\cdot\hat{\vec{x}} = (\hat{\vec{x}} - \Delta\vec{x})\,\mathcal{I}$$

$$\langle\alpha|\, e^{\frac{i\hat{p}\Delta x}{\hbar}}\left(\Delta x\, e^{\frac{-i\hat{p}\Delta x}{\hbar}} + e^{\frac{-i\hat{p}\Delta x}{\hbar}}\cdot\hat{x}\right)|\alpha\rangle = \langle\alpha|\hat{x}|\alpha\rangle + \langle\alpha|\Delta\hat{x}|\alpha\rangle$$

$$\langle\alpha|\mathcal{I}^\dagger \Delta x\, \mathcal{I}|\alpha\rangle + \langle\alpha|\mathcal{I}^\dagger\mathcal{I}\,\hat{x}|\alpha\rangle$$

$$\boxed{\langle\hat{x}\rangle_{\mathcal{I}(\Delta x)|\alpha\rangle} = \langle\hat{x}\rangle_\alpha + \Delta x}$$

El valor de expectación se traslada

Lo podríamos generalizar a 3D como $\rightarrow$ $\langle\hat{\vec{x}}\rangle_{\mathcal{I}(\Delta\vec{x})|\alpha\rangle} = \langle\hat{\vec{x}}\rangle_{|\alpha\rangle} + \Delta\vec{x}$

24.

a)

$$\langle p'|\hat{x}|\alpha\rangle = i\hbar \frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

$$\int dx' \langle p'|x'\rangle\langle x'|\hat{x}|\alpha\rangle$$

$$\int dx' \cdot \frac{1}{\sqrt{2\pi\hbar}} \cdot e^{-i\frac{p'x'}{\hbar}} \langle x'|\alpha\rangle x'$$

$$= \int dx' \cdot \frac{e^{-i\frac{p'x'}{\hbar}}}{\sqrt{2\pi\hbar}} \cdot x' \cdot \langle x'|\alpha\rangle$$

$$\frac{\partial}{\partial p'}\left(\frac{e^{-i\frac{p'x'}{\hbar}}}{\sqrt{2\pi\hbar}}\right) = \frac{e^{-i\frac{p'x'}{\hbar}}}{\sqrt{2\pi\hbar}} \cdot \left(-\frac{ix'}{\hbar}\right) = -\frac{i}{\hbar}\left(\frac{e^{-i\frac{p'x'}{\hbar}}}{\sqrt{2\pi\hbar}} x'\right)$$

$$= \int dx' \left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial p'}\left\{\frac{e^{-i\frac{p'x'}{\hbar}}}{\sqrt{2\pi\hbar}}\right\}\right)\langle x'|\alpha\rangle$$

$$= \int dx' \, i\hbar \frac{\partial}{\partial p'}\left(\langle p'|x'\rangle\right)\langle x'|\alpha\rangle$$

$$= i\hbar \frac{\partial}{\partial p'}\left(\int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle\right)$$

$$\boxed{\langle p'|\hat{x}|\alpha\rangle = i\hbar \frac{\partial}{\partial p'}\langle p'|\alpha\rangle}$$

$$\langle \beta|\hat{x}|\alpha\rangle = \int dp' \, \psi_\beta^*(p') \, i\hbar \frac{\partial}{\partial p'}\psi_\alpha(p')$$

$$\int dp' \langle \beta|p'\rangle\langle p'|\hat{x}|\alpha\rangle =$$

$$\int dp' \langle \beta|p'\rangle \, i\hbar \frac{\partial}{\partial p'}\langle p'|\alpha\rangle =$$

$$\int dp' \, (\langle p' | \beta \rangle)^* \, i\hbar \frac{\partial}{\partial p'} \, \Psi_\alpha(p')$$

$$\boxed{\langle \beta | \hat{x} | \alpha \rangle = \int dp' \, \Psi_\beta^*(p') \, i\hbar \frac{\partial}{\partial p'} \Psi_\alpha(p')}$$

b)

$$e^{\frac{i \hat{x} \Xi}{\hbar}}$$ con Ξ un número con unidades de momento

Tiene pinta de "traslación" finita del momento en una cantidad asociada con Ξ

Podríamos aplicarlo a un estado $|p'\rangle$ pero no se como opera $\Rightarrow$ se lo aplico a un estado $|x'\rangle$ ∴

$$e^{\frac{i \hat{x} \Xi}{\hbar}} |x'\rangle = \sum_{n=0}^{\infty} \left(\frac{i \hat{x} \Xi}{\hbar}\right)^n \frac{1}{n!} |x'\rangle$$

$$\sum_{n=0}^{\infty} \left(\frac{i x' \Xi}{\hbar}\right)^n \frac{1}{n!} |x'\rangle$$

$$e^{\frac{i x' \Xi}{\hbar}} |x'\rangle$$

$\hat{x}^n |x'\rangle$

$x'^n |x'\rangle$

Se lo aplico a $|p'\rangle$

$$= \int dx' \, e^{\frac{i x' \Xi}{\hbar}} |x'\rangle\langle x' | p'\rangle$$

$$= \int dx' \, e^{\frac{i x' \Xi}{\hbar}} |x'\rangle \frac{e^{\frac{i x' p'}{\hbar}}}{\sqrt{2\pi\hbar}}$$

$$= \int dx' \, \frac{e^{\frac{i x'}{\hbar}(\Xi + p')}}{\sqrt{2\pi\hbar}} |x'\rangle = \int dx' \, (\langle x' | \Xi + p'\rangle)(|x'\rangle)$$

$$= \int dx' \, (|x'\rangle\langle x'| \, \Xi + p'\rangle \Rightarrow$$

$$\boxed{e^{\frac{i \hat{x} \Xi}{\hbar}} |p'\rangle = |\Xi + p'\rangle}$$

Es un operador de traslación de momento