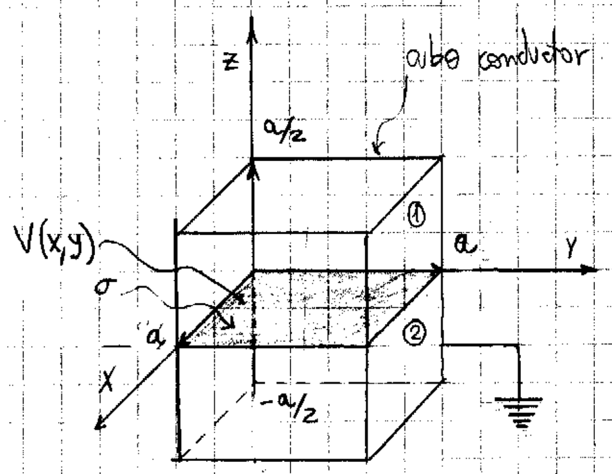
GUÍA 3

Método de Separación de Variables

1.



Dividimos el interior en dos zonas donde vale Laplace

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

proponemos

$$\phi = X(x) \cdot Y(y) \cdot Z(z)$$

$$Z \cdot Y \cdot \frac{\partial^2 X}{\partial x^2} + X \cdot Z \frac{\partial^2 Y}{\partial y^2} + X \cdot Y \cdot \frac{\partial^2 \phi}{\partial z^2} = 0$$

Simetría de reflexión en el plano xy ⇒ ϕ es par en z

$$\frac{Z \cdot Y}{X \cdot Y \cdot Z} \cdot \frac{d^2 X}{dx^2} + \frac{X \cdot Z}{X \cdot Y \cdot Z} \frac{d^2 Y}{dy^2} + \frac{X \cdot Y}{X \cdot Y \cdot Z} \cdot \frac{d^2 Z}{dz^2} = 0$$

$$\underbrace{\frac{1}{X} \frac{d^2 X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y} \cdot \frac{d^2 Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z} \cdot \frac{d^2 Z}{dz^2}}_{+\gamma^2} = 0$$

No pueden ser todos positivos pues sino no puede ser nula la suma

$$\gamma^2 = \alpha^2 + \beta^2$$

$$\frac{1}{Z} \cdot \frac{d^2 Z}{dz^2} = \gamma^2 \qquad\qquad \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = -\alpha^2$$

$$Z'' - \gamma^2 Z = 0 \qquad\qquad X'' + \alpha^2 X = 0$$

$A e^{i\lambda z}$

$A \cdot e^{i\lambda z} i\lambda$

$-A \cdot e^{i\lambda z} \lambda^2$

$$A e^{i\lambda z}(\lambda^2 - \gamma^2) = 0 \qquad\qquad (\lambda^2 + \alpha^2) = 0$$

$$\lambda = \begin{cases} \gamma \\ -\gamma \end{cases} \qquad\qquad \lambda = \begin{cases} i\alpha \\ -i\alpha \end{cases}$$

$$Z \propto e^{\pm \gamma z} = e^{\pm\sqrt{\alpha^2+\beta^2}\, z} \qquad\qquad X \propto e^{\pm i \alpha x}$$

análogamente $Y \propto e^{\pm i \beta y}$

$$\phi = e^{\pm i\alpha x} \cdot e^{\pm i\beta y} \cdot e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

NOTA

Podemos considerarlo como suma de seno y coseno

$$X = A \cdot \cos(\alpha x + \varphi_0)$$
$$A \cdot \cos \alpha x \cdot \cos\varphi_0 - A \cdot \operatorname{sen} \alpha x \cdot \operatorname{sen}\varphi_0$$

$$\phi(x,y,z) = \left[A \cdot \cos(\alpha x) + B \cdot \operatorname{sen}(\alpha x)\right] \cdot \left[C \cdot \cos(\beta y) + D \cdot \operatorname{sen}(\beta y)\right] \left[E \cdot \operatorname{senh}(\gamma z) + F \cdot \cosh(\gamma z)\right]$$

* contornos Región ①

$\phi(x,0,z) = 0 \rightarrow Y = C \cdot \cos(\beta 0) + D \cdot \operatorname{sen}(\beta \cdot 0) = 0 \Rightarrow \boxed{C = 0}$

$0 < x < a \quad 0 < z \leq a/2$

$\phi(x,a,z) = 0 \rightarrow Y = 0 = D \cdot \operatorname{sen}(\beta \cdot a) \Rightarrow \beta \cdot a = n \cdot \pi \rightarrow \boxed{\beta = n\frac{\pi}{a}}$

$\phi(0,y,z) = 0 \longrightarrow X = 0 = A \cdot \cos(\alpha \cdot 0) + B \cdot \operatorname{sen}(\alpha \cdot 0) \Rightarrow \boxed{A = 0}$

$0 < y < a \quad 0 < z \leq a/2$

$\phi(a,y,z) = 0 \longrightarrow X = 0 = B \cdot \operatorname{sen}(\alpha a) \qquad \alpha \cdot a = m \cdot \pi \rightarrow \boxed{\alpha = m\frac{\pi}{a}}$

$\phi(x,y,a/2) = 0 \quad \rightarrow \quad Z=0 = E \cdot \operatorname{senh}(\gamma z) + \underbrace{F}_{=0} \cdot \cosh(\gamma z)$ nunca nulo

$0<x<a, \ 0<y<a$

$= E \cdot \operatorname{senh}\left[\gamma\left(\frac{a}{2} - z\right)\right] = 0$ en $z = \frac{a}{2}$

Un modo de hacer esto es desplazar el argumento y el otro considerar E, F no nulos.

* Región ②

Solo cambia la parte en z

$\phi(x,y,-a/2) = 0 \qquad Z=0 = E \cdot \operatorname{senh}\left[\gamma\left(\frac{a}{2}+z\right)\right] = 0$ en $z = -\frac{a}{2}$

$0<x<a, \ 0<y<a$

continuidad en $z=0$

$E \cdot \operatorname{senh}\left(\gamma \cdot \frac{a}{2}\right) = E \cdot \operatorname{senh}\left(\gamma \cdot \frac{a}{2}\right)$

No me preocupo por el coeficiente E

$$\boxed{\begin{aligned} \phi^{1}_{mn}(x,y,z) &= A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi}{a}x\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) \cdot \operatorname{senh}\left[\left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)\left(\frac{a}{2} - z\right)\right] \\ \phi^{2}_{mn}(x,y,z) &= A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi}{a}x\right) \cdot \operatorname{sen}\left(\frac{n\pi}{a}y\right) \cdot \operatorname{senh}\left[\left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)\left(\frac{a}{2} + z\right)\right] \end{aligned}}$$

* Salto del campo

$$\left[\frac{\partial\phi^1}{\partial z} + \frac{\partial\phi^2}{\partial z}\right]_{z=0} = -\sum_{n,m=1}^{\infty} A_{mn} \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) \cdot \cosh\left[\left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)\frac{a}{2}\right] \cdot \left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)(-)$$

$$+ \sum_{n,m=1}^{\infty} A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) \cdot \cosh\left[\left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)\frac{a}{2}\right]\left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)$$

$$2\sum_{n,m=1}^{\infty} A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) \cdot \cosh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,\frac{a}{2}\right) \cdot \frac{\pi}{a}\sqrt{n^2+m^2} = 4\pi\sigma$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) \cdot \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right) = \frac{2\sigma \cdot a}{\sqrt{n^2+m^2}}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) = \frac{2a\,\sigma(x,y)}{\sqrt{n^2+m^2}} \cdot \frac{1}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

* Ortogonalidad

σ sea constante

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^a A_{mn} \cdot \operatorname{sen}\left(\frac{m\pi x}{a}\right) \operatorname{sen}\left(\frac{m'\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{n\pi y}{a}\right) dx = \frac{2a\sigma}{\sqrt{n^2+m^2} \cdot \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \int_0^a \operatorname{sen}\left(\frac{m'\pi x}{a}\right) \cdot dx$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \operatorname{sen}\left(\frac{n\pi y}{a}\right) A_{mn} \cdot \underbrace{\int_0^a \operatorname{sen}\left(\frac{m\pi x}{a}\right) \cdot \operatorname{sen}\left(\frac{m' \pi x}{a}\right) \cdot dx}_{\int_0^a \operatorname{sen}^2 \ \delta_{mm'}} =$$

$$\sum_{n=1}^{\infty} \operatorname{sen}\left(\frac{n\pi y}{a}\right) A_{m'n} \cdot \int_0^a \operatorname{sen}^2\left(\frac{m\pi x}{a}\right) \cdot dx$$

$u = \frac{m\pi x}{a}$

$du = \frac{m\pi}{a} dx$

$\frac{a}{m\pi} \cdot \int_0^{m\pi} \operatorname{sen}^2 u \, du = a/2$

$$\sum_{n=1}^{\infty}\int_0^a \operatorname{sen}\left(\frac{n\pi y}{a}\right) A_{m'n} \operatorname{sen}\left(\frac{n'\pi y}{a}\right) \frac{a}{2}\, dy = \frac{2a\sigma}{\sqrt{n^2+m^2} \cosh\left(\frac{\pi\sqrt{n^2+m^2}}{2}\right)} \int_0^a dy \cdot \operatorname{sen}\left(\frac{m\pi y}{a}\right) \int_0^a dx \cdot \operatorname{sen}(\ldots$$

$$\frac{a}{2} \cdot \frac{a}{2} A_{mn} =$$

$$A_{mn} = \frac{8\ \sigma}{a\sqrt{n^2+m^2}\ \cosh\left(\frac{\pi}{Z}\sqrt{n^2+m^2}\right)} \int_0^a \operatorname{sen}\left(\frac{m\pi y}{a}\right) dy \int_0^a dx \ \operatorname{sen}\left(\frac{n\pi x}{a}\right)$$

$$\frac{a}{m\pi}\int_0^{m\pi} \operatorname{sen} u\, du \cdot \frac{a}{n\pi}\int_0^{n\pi} \operatorname{sen} u\, du$$

$$A_{mn} = \frac{8\sigma}{\cancel{a}\sqrt{n^2+m^2}\cosh\left(\frac{\pi}{Z}\sqrt{n^2+m^2}\right)} \frac{a^{\cancel{2}}}{mn\,\pi^2}\left(-\cos u \Big|_0^{m\pi}\right)\left(-\cos u\Big|_0^{n\pi}\right)$$

$$+\left[(-1)^n - 1\right]\left[(-1)^n - 1\right]$$

$$\boxed{A_{mn} = \frac{8\sigma a}{\sqrt{n^2+m^2}\cdot\pi^2\cosh\left(\frac{\pi\sqrt{n^2+m^2}}{Z}\right) m\, n}\left[(-1)^n - 1\right]^2}$$

* Algún cálculo extra

$$0 = E\,\operatorname{senh}\left(\gamma\frac{a}{2}\right) + F\cosh\left(\gamma\frac{a}{2}\right) = \frac{E}{2}e^{\gamma\frac{a}{2}} - \frac{E}{2}e^{-\gamma\frac{a}{2}} + \frac{F}{2}e^{\gamma\frac{a}{2}} + \frac{F}{2}e^{-\gamma\frac{a}{2}}$$

$$\frac{1}{2}(E+F)e^{\gamma\frac{a}{2}} + \frac{1}{2}(F-E)e^{-\gamma\frac{a}{2}}$$

$$E = -F\frac{\cosh(\gamma a/2)}{\operatorname{senh}(\gamma a/2)} \rightarrow Z = -F\frac{\operatorname{senh}(\gamma z)\cosh(\gamma a/2)}{\operatorname{senh}(\gamma a/2)} + F\cosh(\gamma z)\ldots$$

$$F\left(\frac{-\operatorname{senh}(\gamma z)\cosh(\gamma a/2) + \cosh(\gamma z)\ldots}{\operatorname{senh}(\gamma a/2)}\right)$$

$$Z(z=0) = F\cosh(0)$$

$$0 = -E\,\operatorname{senh}\left(\gamma\frac{a}{2}\right) + F\cosh\left(\gamma\frac{a}{2}\right)$$

$$E = F\frac{\cosh(\gamma a/2)}{\operatorname{senh}(\gamma a/2)}$$

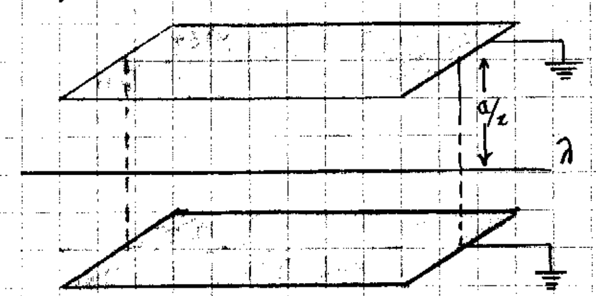
$$Z = F\left(\frac{\operatorname{senh}(\gamma z)\cosh(\gamma a/2) + \cosh(\gamma z)\operatorname{senh}\ldots}{\operatorname{senh}(\gamma a/2)}\right)$$

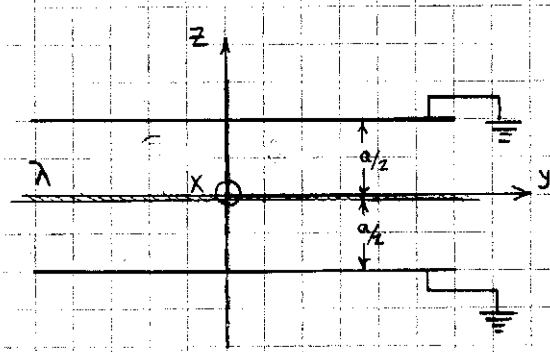
$$Z(z=0) = F\cosh(0)$$

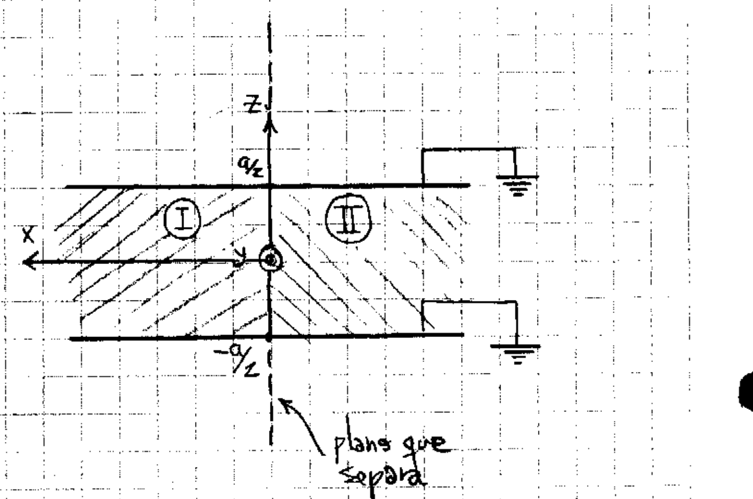
$$F\cosh 0 = F\cosh 0$$

no hace falta preocuparse por el coeficiente F

2. (a)







$\lambda = \frac{Q}{L}$ (carga por unidad de longitud)

$$\rho(\vec{r}) = \delta(z-0).\delta(x-0).\lambda$$

$$\rho(\vec{r}) = \lambda.\delta(z).\delta(x)$$

$$\sigma(z) = \lambda.\delta(z)$$

* Simetrías

Traslación en $\hat{y} \rightarrow \vec{E} \neq \vec{E}(y)$

Reflexión en $zy \rightarrow$

Reflexión en $xy \rightarrow$

Reflexión en $xz \rightarrow E_y = 0$

$$-\frac{\partial\phi}{\partial y} = 0$$

$$\phi \neq \phi(y)$$

propongo

$$\phi = X.Z.Y$$

$Y = (\text{cte}) \rightarrow$ por simetría

parece más apropiados trig

parece más factible exp.

* en I y II

$$\nabla^2\phi = 0$$

$$\cancel{\frac{Z}{XZ}}\frac{\partial^2\phi}{\partial x^2} + \cancel{\frac{X}{XZ}}\frac{\partial^2\phi}{\partial z^2} = 0$$

$$+k^2 \quad -k^2 \quad = 0$$

$$X'' - k^2 X = 0 \qquad X = e^{\pm kx}$$

$$Z'' + k^2 Z = 0 \qquad Z = e^{\pm ikz}$$

* Región I

$$\phi(x,z) = \left[A.\cos(kz) + B.\text{sen}(kz)\right].\left[C.e^{kx} + D.e^{-kx}\right]$$

$x \rightarrow +\infty \Rightarrow \phi = 0 \qquad 0 = C.e^{kx} + D.e^{-kx} \rightarrow \boxed{C=0}$

$z = a/2 \Rightarrow \phi = 0 \qquad 0 = A.\cos\left(\frac{ka}{2}\right) + B.\text{sen}\left(\frac{ka}{2}\right) \rightarrow B=0$

$0 < x < +\infty$

$0 = A.\cos\left(\frac{ka}{2}\right)$ con $\frac{ka}{2} = (2m-1)\frac{\pi}{2}$

$$\boxed{k = (2m-1)\frac{\pi}{a}}$$

Conviene usar coseno pues $\cos\left(\frac{a}{2}\right) = \cos\left(-\frac{a}{2}\right)$ como es requerido por la simetría

$$\phi_{I_m}(x,z) = G_I \cdot \cos(k_m z) \cdot e^{-k_m x}$$

* Región (II)

$$x \to -\infty \Rightarrow \phi = 0 \qquad 0 = C \cdot e^{kx} + D \cdot e^{-kx} \Rightarrow \boxed{D=0}$$

$$z = \pm a/2 \Rightarrow \phi = 0 \qquad 0 = A \cdot \cos\left(\frac{ka}{2}\right) \qquad \text{con} \quad k = (2m-1)\frac{\pi}{a}$$

$-\infty < x < 0$

$$\phi_{II_m}(x,z) = G_{II} \cdot \cos(k_m z) \cdot e^{k_m x}$$

* Continuidad del Potencial en el plano YZ $(x=0)$

$$\phi_I(0,z) = \phi_{II}(0,z)$$

$$\sum_{m=1}^{\infty} G_I^m \cos(k_m z) = \sum_{m=1}^{\infty} G_{II}^m \cdot \cos(k_m z) \Rightarrow G_I^m = G_{II}^m \equiv C_m$$

$$\phi_I = \sum_{m=1}^{\infty} C_m \cdot \cos\left[(2m-1)\frac{\pi}{a} z\right] \cdot e^{-(2m-1)\frac{\pi}{a}x}$$

$$\phi_{II} = \sum_{m=1}^{\infty} C_m \cdot \cos\left[(2m-1)\frac{\pi}{a} z\right] \cdot e^{(2m-1)\frac{\pi}{a}x}$$

* Salto del campo $\vec{E}$

$$\left[\frac{\partial \phi_{II}}{\partial x} - \frac{\partial \phi_I}{\partial x}\right]_{x=0} = 4\pi\sigma$$

$$\sum_{m=1}^{\infty} \Big\{ C_m \cdot \cos\left[(2m-1)\frac{\pi}{a} z\right] \cdot e^{(2m-1)\frac{\pi}{a}x} \cdot (2m-1)\frac{\pi}{a}$$

$$- C_m \cdot \cos\left[(2m-1)\frac{\pi}{a} z\right] \cdot e^{-(2m-1)\frac{\pi}{a}x} \cdot (2m-1)\frac{\pi}{a} \cdot (-) \Big\}\Big|_{x=0} = 4\pi \cdot \lambda \cdot \delta(z)$$

$$\sum_{m=1}^{\infty} 2 C_m \cdot \cos\left[(2m-1) \cdot \frac{\pi}{a} z\right] \cdot (2m-1) \cdot \frac{\pi}{a} = 4\pi \cdot \lambda \cdot \delta(z)$$

$$\sum_{m=1}^{\infty} C_m \cdot \cos(k_m z) \cdot k_m \cdot 2 = 4\pi\lambda\,\delta(z)$$

* Ortogonalidad

$$\int_{-a/2}^{a/2} \sum_{m=1}^{\infty} 2 \cdot C_m \cdot \cos\left[(2m-1)\frac{\pi}{a} z\right] \cdot \cos\left[(2m'-1)\frac{\pi}{a} z\right] \cdot k_m = \int_{-a/2}^{a/2} 4\pi\,\lambda\,\delta(z) \cdot \cos(k_{m'}$$

$$\sum_{m=1}^{\infty} 2k_m C_m \underbrace{\int_{-\frac{a}{2}}^{+\frac{a}{2}} \cos(k_m z)\cdot\cos(k_{m'} z)\,dz}_{\int \cos^2 \quad \delta_{mm'}} = 4\pi\lambda \int_{-\frac{a}{2}}^{+\frac{a}{2}} \delta(z)\cos(k_m z)\,dz$$

$$= 4\pi\cdot\lambda\cdot 1$$

$$\frac{a}{\pi(2m-1)}\int_{-(2m-1)\frac{\pi}{2}}^{+(2m-1)\frac{\pi}{2}} \cos^2 u\,du \qquad \left(\frac{2m-1}{2}\right)\pi$$

recordad

$$\boxed{\int_{-a/2}^{+a/2} \cos(k_m z)\cos(k_{m'} z) = \delta_{mm'}\cdot\frac{a}{2}}$$

$$k_m = \frac{2m\pi x}{a}$$

$$(2m-1)\frac{\pi}{a} z = u$$

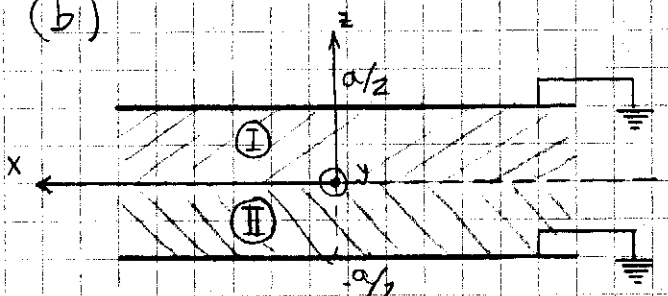
$$(2m-1)\frac{\pi}{a} dz = du$$

$$\cancel{2} k_m C_m \cdot \frac{a}{\cancel{\pi}\cancel{(2m-1)}}\cdot\frac{\cancel{(2m-1)}\cancel{\pi}}{\cancel{2}} = 4\pi\lambda$$

$$C_m = \frac{4\pi\lambda}{\cancel{a}}\frac{\cancel{a}}{(2m-1)\cancel{\pi}} = \frac{4\lambda}{(2m-1)}$$

$$\boxed{\begin{aligned}\phi_I &= \sum_{m=1}^{\infty}\frac{4\lambda}{(2m-1)}\cos\left[(2m-1)\frac{\pi}{a}z\right]\cdot e^{-\left[(2m-1)\frac{\pi}{a}x\right]}\\ \phi_{II} &= \sum_{m=1}^{\infty}\frac{4\lambda}{(2m-1)}\cos\left[(2m-1)\frac{\pi}{a}z\right]\cdot e^{\left[(2m-1)\frac{\pi}{a}x\right]}\end{aligned}}$$

(b)



EL análisis de simetrías y forma del potencial es el mismo dado que solo subdividimos el espacio de otro modo

NOTA
si separo con el plano XY ⇒ atraviesa carga en $\hat{z}$ ⇒ tomo hiperbólicas para $\hat{z}$

Propongo $\phi = X.Z.Y$

$\to Y = (cte)$ por la simetría

$\to$ se recomiendan exponenciales reales [atravesamos densidad de carga]

$\to$ se recomiendan exponenciales complejas [no hay contornos]

* Región I

$$\phi(x,z) = \left[A_k\, e^{ikx}\right]\left[C_k\cdot e^{kz} + D_k\cdot e^{-kz}\right]$$

Como no hay contornos en $X \to$ no hay discretización de $k \to$ tomo e^{ikx} para que la integral converja

$$\phi(x, z=a/2) = 0$$

$$0 = C\cdot e^{k\frac{a}{2}} + D\cdot e^{-k\frac{a}{2}}$$

optar por desplazar a ojo el senh o bien usar ambos exp [son equivalentes] se puede

$$0 = C\cdot \operatorname{senh}\left[k\left(\frac{a}{2}-z\right)\right] \iff z = \frac{a}{2}$$

$X(x\to\infty) = 0$

Pero no hay requisito en $k \to$ debo tener una integral

$$\phi_I(x,z) = \int_{-\infty}^{+\infty} A_k^I\, e^{ikx}\operatorname{senh}\left[k\left(\frac{a}{2}-z\right)\right]dk$$

* Región (II)

$$\phi(x,z) = [A_k . e^{ikx}] [C_k e^{kz} + D_k e^{-kz}]$$

$$\phi(x, z = -a/2) = 0$$

$$0 = C e^{-ka/2} + D.e^{ka/2}$$

$$0 = C . \operatorname{senh}\left[k\left(\frac{a}{2}+z\right)\right]$$

$$\Leftrightarrow \quad z = -\frac{a}{2}$$

meto todo en una constante

$$\phi_{II}(x,z) = \int_{-\infty}^{+\infty} A_k^{II} . e^{ikx} . \operatorname{senh}\left[k\left(\frac{a}{2}+z\right)\right] . dk$$

* Continuidad del ϕ en el plano XY $(z=0)$

$$\phi_I(x,0) = \phi_{II}(x,0)$$

$$\int_{-\infty}^{+\infty} A_k^{I} . e^{ikx} . \operatorname{senh}\left(\frac{ka}{2}\right) dk = \int_{-\infty}^{+\infty} A_k^{II} . e^{ikx} \operatorname{senh}\left(\frac{ka}{2}\right) dk$$

$$\Rightarrow \quad A_k^{I} = A_k^{II} \equiv A_k$$

* Salto del campo $\vec{E}$

$$\rho(\vec{r}) = \delta(z) . \delta(x) . \lambda$$

$$\left[\frac{\partial \phi_{II}}{\partial z} - \frac{\partial \phi_I}{\partial z}\right]_{z=0} = 4\pi . \delta(x) . \lambda \qquad \sigma(x) = \delta(x) . \lambda$$

$$\int_{-\infty}^{+\infty} A_k . e^{ikx} \cosh\left[k\left(\frac{a}{2}+z\right)\right] . k . dk - \int_{-\infty}^{+\infty} A_k . e^{ikx} . \cosh\left[k\left(\frac{a}{2}-z\right)\right](-k) . dk = 4\pi . \delta(x) . \lambda$$

$$\int_{-\infty}^{+\infty} 2 . A_k . e^{ikx} . \cosh\left(\frac{ka}{2}\right) . k . dk = 4\pi . \delta(x) . \lambda$$

$$\int_{-\infty}^{+\infty} A_k . e^{ikx} . \cosh\left(\frac{ka}{2}\right) k . dk = 2\pi \lambda \, \delta(x)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A_k . e^{ikx} \cosh\left(\frac{ka}{2}\right) k . dk . e^{-ik'x} . dx = \int_{-\infty}^{+\infty} e^{-ik'x} . 2\pi\lambda . \delta(x) . dx$$

$$\int_{-\infty}^{+\infty} A_k . \cosh\left(\frac{ka}{2}\right) k \, dk \int_{-\infty}^{+\infty} e^{ix(k-k')} . dx = 2\pi\lambda$$

$$\int_{-\infty}^{+\infty} A_k . \cosh\left(\frac{ka}{2}\right) k . dk . 2\pi . \delta(k-k') = 2\pi\lambda$$

$$A_k . \cosh\left(\frac{ka}{2}\right) . k . \cancel{2\pi} = \cancel{2\pi} \lambda$$

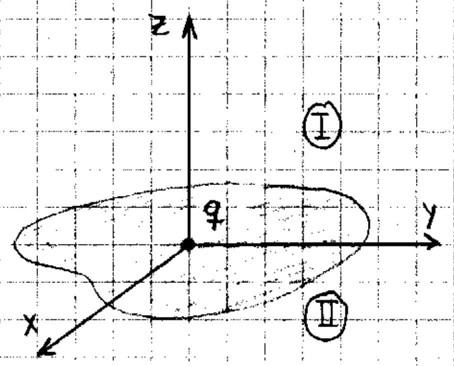
$$A_k = \frac{\lambda}{k \cdot \cosh\left(\frac{ka}{2}\right)}$$

$$\boxed{\phi_I(x,z) = \int_{-\infty}^{+\infty} \frac{\lambda}{k \cosh\left(\frac{ka}{2}\right)} \, e^{ikx} \, \operatorname{senh}\left(k\left[\frac{a}{2} - z\right]\right) dk}$$

$$\boxed{\phi_{II}(x,z) = \int_{-\infty}^{+\infty} \frac{\lambda}{k \cosh\left(\frac{ka}{2}\right)} \, e^{ikx} \, \operatorname{senh}\left(k\left[\frac{a}{2} + z\right]\right) dk}$$

3.

a.



Carga puntual en el origen ; totalmente simétrico

reflexión en XY
" " ZX
" " ZY

$$\rho(\vec{r}) = \underbrace{q\,\delta(x)\,\delta(y)}_{\sigma(x,y)}\,\delta(z)$$

Tomo $z=0$ plano con $\sigma \Rightarrow$

$$\begin{cases} X, Y & \text{exp. complejas} \\ Z & \text{exp. reales} \end{cases} \rightarrow \begin{cases} z\to\infty & \phi=0 \\ z\to-\infty & \phi=0 \end{cases} \rightarrow \begin{cases} e^{-\gamma z} & \text{(región I)} \\ e^{\gamma z} & \text{(región II)} \end{cases}$$

$$\phi_I(x,y,z) = A^I_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, e^{-\gamma z} \qquad \text{con } \gamma = \sqrt{\alpha^2+\beta^2}$$

* Región (I)

$$\phi_I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A^I_{\alpha\beta}\, e^{i\alpha x} e^{i\beta y}\, e^{-\gamma z}\, d\alpha\, d\beta$$

* Región (II)

$$\phi_{II} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A^{II}_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, e^{\gamma z}\, d\alpha\, d\beta$$

* Continuidad del ϕ en el plano $z=0$

$$\phi_I(x,y,0) = \phi_{II}(x,y,0)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A^I_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, d\alpha\, d\beta = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A^{II}_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, d\alpha\, d\beta$$

$$A^I_{\alpha\beta} = A^{II}_{\alpha\beta} \equiv A_{\alpha\beta} \quad \Rightarrow$$

$$\phi(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, e^{-\sqrt{\alpha^2+\beta^2}\,|z|}\, d\alpha\, d\beta$$

* Salto en el campo $\vec{E}$

$$\left.\frac{\partial\phi_{II}}{\partial z} - \frac{\partial \phi_I}{\partial z}\right|_{z=0} = 4\pi\sigma$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, \underbrace{e^{+\sqrt{\alpha^2+\beta^2}\,z}}_{=1}\,\sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta$$

$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, \underbrace{e^{-\sqrt{\alpha^2+\beta^2}\,z}}_{=1}\,\sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta = 4\pi q\,\delta(x)\,\delta(y)$$

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{\alpha\beta}\, e^{i\alpha x}\cdot e^{i\beta y}\cdot\sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta = 4\pi\cdot q\cdot \delta(x)\cdot\delta(y)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2\, e^{-i\alpha' x}\cdot e^{-i\beta' y}\, dx\, dy\, A_{\alpha\beta}\, e^{i\alpha x}\cdot e^{i\beta y}\cdot\sqrt{\alpha^2+\beta^2}\cdot d\alpha\cdot d\beta =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 4\pi\cdot q\cdot\delta(x)\cdot\delta(y)\cdot e^{-i\alpha' x}\cdot e^{-i\beta' y}\, dx\cdot dy$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2\left(\int_{-\infty}^{+\infty} e^{ix(\alpha-\alpha')}\, dx\right)\left(\int_{-\infty}^{+\infty} e^{iy(\beta-\beta')}\, dy\right) A_{\alpha\beta}\sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta = 4\pi q$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2\cdot 2\pi\cdot\delta(\alpha-\alpha')\; 2\pi\cdot\delta(\beta-\beta')\, A_{\alpha\beta}\cdot\sqrt{\alpha^2+\beta^2}\cdot d\alpha\, d\beta = 4\pi q$$

$$\cancel{8}^{\,2}\,\pi^{\cancel{2}}\, A_{\alpha'\beta'}\sqrt{\alpha'^2+\beta'^2} = \cancel{4\pi}\, q$$

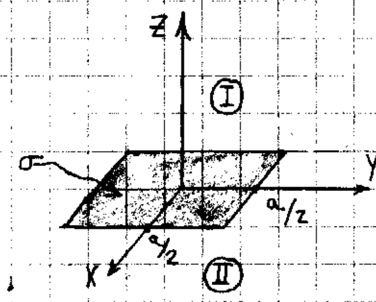
saco las primas

$$A_{\alpha\beta} = \frac{q}{2\sqrt{\alpha^2+\beta^2}\cdot\pi}$$

$$\boxed{\phi(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{q}{2\pi\sqrt{\alpha^2+\beta^2}}\, e^{i\alpha x}\cdot e^{i\beta y}\, e^{-\sqrt{\alpha^2+\beta^2}\,|z|}\, d\alpha\cdot d\beta}$$

▲ Expresión válida para ambas regiones

b.



Supongo una σ uniforme

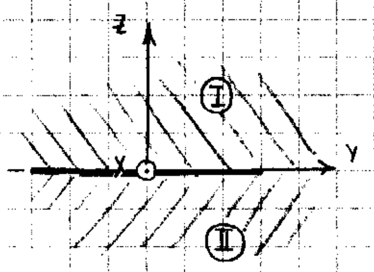
$$\rho(\vec{r}) = \sigma \cdot \delta(z) \qquad \begin{matrix} -a/2 < x < a/2 \\ -a/2 < y < a/2 \end{matrix}$$

Tiene simetría de reflexión en XY
Tiene simetría de reflexión en ZX
Tiene simetría de reflexión en ZY

Propongo $\phi = X.Y.Z$

- Z → Como atravieso la σ utilizaré exp. reales
- X, Y → exp. complejas

$$\phi = e^{\pm i\alpha x} \cdot e^{\pm i\beta y} \cdot e^{\pm \gamma z} \qquad \text{con } \gamma^2 = \alpha^2 + \beta^2$$



* Región I

$$\left\{ \begin{matrix} z \to +\infty \\ x, y \text{ cualquiera} \end{matrix} \right. \quad \phi = 0 \Rightarrow Z = 0 = \underbrace{E.e^{\gamma z}}_{=0} + F.e^{-\gamma z}$$

$$Z = F.e^{-\gamma z}$$

Por la simetría de reflexión podría tomar cosenos para X e Y, pero la integral se complicaría

$$\phi_I = A^{\pm}_{\alpha\beta} \cdot e^{i\alpha x} \cdot e^{i\beta y} \cdot e^{-\sqrt{\alpha^2+\beta^2}\,.z}$$

Como no hay discretización en α, β [para $\hat{x}$ e $\hat{y}$ no hay contornos] $\Rightarrow$

$$\phi_I(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{\alpha\beta} \cdot e^{i\alpha x} \cdot e^{i\beta y} \cdot e^{-\sqrt{\alpha^2+\beta^2}\,.z} \cdot d\alpha \cdot d\beta$$

Obtenemos una integral de fourier

* Región II

$$\phi_{II} = A^{II}_{\alpha\beta} \cdot e^{i\alpha x} \cdot e^{i\beta y} \cdot e^{\sqrt{\alpha^2+\beta^2}\,.z}$$

$$\phi_{II}(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A^{II}_{\alpha\beta} \cdot e^{i\alpha x} \cdot e^{i\beta y} \cdot e^{\sqrt{\alpha^2+\beta^2}\,.z} \cdot d\alpha \cdot d\beta$$

* Continuidad en el plano XY $\quad (z=0)$

$$\phi_I(x,y,0) = \phi_{II}(x,y,0)$$

$$\int_{-\infty}^{+\infty}\int A^{I}_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y} \cdot d\alpha\, d\beta = \int_{-\infty}^{+\infty}\int A^{II}_{\alpha\beta}\, e^{i\alpha x} \cdot e^{i\beta y}\, d\alpha \cdot d\beta$$

$$A^{I}_{\alpha\beta} = A^{II}_{\alpha\beta} \equiv A_{\alpha\beta}$$

* Salto en el campo al atravesar el Cuadrado

Son $-\frac{a}{2} < x < \frac{a}{2}$; $-\frac{a}{2} < y < \frac{a}{2}$ $\Rightarrow$

$$\left.\frac{\partial \phi_{II}}{\partial z} - \frac{\partial \phi_{I}}{\partial z}\right|_{z=0} = 4\pi\sigma$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, \underbrace{e^{\sqrt{\alpha^2+\beta^2}\, z}}_{=1}\, \sqrt{\alpha^2+\beta^2} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\, \underbrace{e^{\sqrt{\alpha^2+\beta^2}\, z}}_{=1}\sqrt{\alpha^2+\beta^2}$$

$$= 4\pi\sigma$$

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\sqrt{\alpha^2+\beta^2} = 4\pi\sigma = 4\pi\begin{cases}\sigma & \text{si } \begin{cases} x\in\left(-\frac{a}{2},\frac{a}{2}\right)\\ y\in\left(-\frac{a}{2},\frac{a}{2}\right)\end{cases}\\ 0 \end{cases}$$

* Ortogonalidad y calculo de $A_{\alpha\beta}$

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\alpha' x}\, dx\, d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\alpha x}\, e^{i\beta y}\sqrt{\alpha^2+\beta^2} = \int_{-\infty}^{+\infty} 4\pi\sigma\, dx\, e^{-i\alpha' x}$$

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{ix(\alpha-\alpha')}\, dx\right) d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\beta y}\sqrt{\alpha^2+\beta^2} = 4\pi\sigma\int_{-a/2}^{a/2} dx\, e^{-i\alpha' x}$$

integral inmediata

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2\pi\,\delta(\alpha-\alpha')\, d\alpha\, d\beta\, A_{\alpha\beta}\, e^{i\beta y}\sqrt{\alpha^2+\beta^2} = 4\pi\sigma\,\frac{e^{i\alpha'\frac{a}{2}} - e^{-i\alpha'\frac{a}{2}}}{i\alpha'}$$

$$2\int_{-\infty}^{+\infty} 2\pi\, A_{\alpha'\beta}\, e^{i\beta y}\sqrt{\alpha'^2+\beta^2}\, d\beta = 4\pi\sigma\,\frac{2}{\alpha'}\,\text{sen}\left(\frac{\alpha' a}{2}\right)$$

$$2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\beta' y}\, dy\, 2\pi\, A_{\alpha'\beta}\, e^{i\beta y}\sqrt{\alpha'^2+\beta^2}\, d\beta = 4\pi\int_{-\infty}^{+\infty}\sigma\,\frac{2}{\alpha'}\, e^{-i\beta' y}\, dy\ \text{sen}\left(\frac{\alpha' a}{2}\right)$$

$$2\int_{-\infty}^{+\infty}\underbrace{\left(\int_{-\infty}^{+\infty} e^{iy(\beta-\beta')}\, dy\right)}_{2\pi\,\delta(\beta-\beta')} 2\pi\, A_{\alpha'\beta}\sqrt{\alpha'^2+\beta^2}\, d\beta = \frac{8\pi}{\alpha'}\,\text{sen}\left(\frac{\alpha' a}{2}\right)\int_{-a/2}^{a/2}\sigma\, e^{-i\beta' y}\, dy$$

$$2\cdot 4\pi^2 A_{\alpha'\beta'}\sqrt{\alpha'^2+\beta'^2} = \frac{8\pi}{\alpha'}\,\text{sen}\left(\frac{\alpha' a}{2}\right)\frac{e^{i\beta'\frac{a}{2}} - e^{-i\beta'\frac{a}{2}}}{i\beta'} = \frac{16\pi}{\alpha'\beta'}\,\text{sen}\left(\frac{\alpha' a}{2}\right)\text{sen}\left(\frac{\beta' a}{2}\right)$$

$$A_{\alpha\beta} = \frac{2\ \text{sen}(\alpha\, a/2)\cdot\text{sen}(\beta\, a/2)}{\pi\sqrt{\alpha^2+\beta^2}\,\alpha\,\beta}$$

$$\boxed{\phi(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{2\ \text{sen}\left(\alpha\frac{a}{2}\right)\text{sen}\left(\beta\frac{a}{2}\right)}{\pi\,\alpha\,\beta\,\sqrt{\alpha^2+\beta^2}}\, e^{i\alpha x}\, e^{i\beta y}\, e^{-\sqrt{\alpha^2+\beta^2}\,|z|}\, d\alpha\, d\beta}$$

* CA

$$\int_{-a/2}^{a/2} e^{-i\alpha x}\, dx =$$

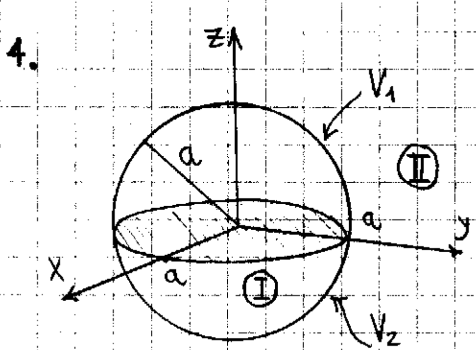
$u = -i\alpha x$, $du = -i\alpha\, dx$] sustitución

$$-\frac{1}{i\alpha}\int_{\frac{i\alpha a}{2}}^{-\frac{i\alpha a}{2}} e^{u}\, du =$$

$$\frac{e^{\frac{i\alpha a}{2}} - e^{-\frac{i\alpha a}{2}}}{i\alpha} =$$

$$\boxed{2\,\frac{\text{sen}\left(\frac{\alpha a}{2}\right)}{\alpha}}$$

4.



Es conveniente situar el eje azimutal en $\hat{z}$ $\Rightarrow$ hay simetría en φ de rotación ; por reflexión $E\varphi = 0$ $\Downarrow$ $\frac{\partial \phi}{\partial \varphi} = 0$

$$\Rightarrow \quad Q = cte. \quad \Rightarrow$$

$$\phi = R(r) \cdot \Theta(\theta)$$

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

$\nabla^2 \phi = 0$ requiere dos regiones ; así dejamos $\rho(\vec{r})$ solo en los contornos

Región I : $r<a$; Región II : $r>a$

* Región I :

$$\phi(r=0, z=0) < \infty \quad \Rightarrow \quad B_l = 0$$

$$\phi_I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l \cdot P_l(\cos\theta) \qquad r<a$$

* Región II :

$$\phi(r\to\infty) < \infty \quad \Rightarrow \quad A_l = 0$$

$$\phi_{II}(r,\theta) = \sum_{l=0}^{\infty} B_l \cdot r^{-l-1} \cdot P_l(\cos\theta) \qquad r>a$$

* Continuidad de ϕ en $r=a$

$$\phi_I(a,\theta) = \phi_{II}(a,\theta)$$

$$\sum_{l=0}^{\infty} A_l \cdot a^l \cdot P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l \cdot a^{-l-1} \cdot P_l(\cos\theta)$$

Como $P_l(\cos\theta)$ son ortogonales $\Rightarrow$ deben ser iguales término a término

$$A_l \cdot a^l = B_l \cdot a^{-l-1}$$

$$A_l \cdot \frac{a^l}{a^{-l-1}} = B_l$$

$$\boxed{B_l = A_l \cdot a^{2l+1}}$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{+1} f(x) \cdot P_l(x) \cdot dx$$

$x = \cos\theta$
$dx = -\operatorname{sen}\theta \cdot d\theta$

$$A_l = \frac{2l+1}{2} \cdot \int_{\pi}^{0} P_l(\cos\theta) \cdot \operatorname{sen}\theta \cdot d\theta \, f(\cos\theta)$$

$$A_l = \frac{2l+1}{2} \int_0^{\pi} P_l(\cos\theta) \cdot \operatorname{sen}\theta \cdot d\theta \, f(\cos\theta)$$

$$\frac{2l+1}{2} \int_0^{\pi} P_{l'}(\cos\theta) \cdot P_l(\cos\theta) \cdot \operatorname{sen}\theta \cdot d\theta = \delta_{ll'}$$

* Ortogonalidad y V en el contorno

$$V(a,\theta) = \sum_{l=0}^{\infty} A_l \cdot a^l \cdot P_l(\cos\theta)$$

$$\int_0^{\pi} V(a,\theta) \left(\frac{2l+1}{2}\right) P_{l'}(\cos\theta) \operatorname{sen}\theta \, d\theta = \int_0^{\pi} \left(\frac{2l+1}{2}\right) \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) P_{l'}(\cos\theta) \operatorname{sen}\theta \cdot d\theta$$

$$\left(\frac{2l+1}{2}\right) \cdot \left[\int_0^{\pi/2} V_1 \cdot P_{l'}(\cos\theta) \operatorname{sen}\theta \, d\theta + \int_{\pi/2}^{\pi} V_2 \underbrace{}_{P_{l'}(\cos\theta)\operatorname{sen}\theta \, d\theta} \right. = \sum_{l=0}^{\infty} A_l a^l \cdot \frac{2l+1}{2} \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \cdot \operatorname{sen}\theta \cdot d\theta$$

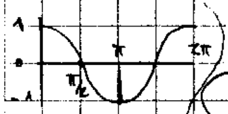
$$\left[\int_0^{\pi/2} V_1 . P_\ell(\cos\theta)\,\text{sen}\theta\, d\theta + \int_{\pi/2}^{\pi} V_2 . P_\ell(\cos\theta)\,\text{sen}\theta\, d\theta\right] . \left(\frac{2\ell+1}{2}\right) = A_\ell\, a^\ell$$

$$\left[-\int_1^0 V_1 . P_\ell(x)\, dx - \int_0^{-1} V_2\, P_\ell(x)\, dx\right] . \left(\frac{2\ell+1}{2a^\ell}\right) = A_\ell$$

$\cos\theta = x$

$-\text{sen}\theta . d\theta = dx$

$$\left(V_1 \int_0^1 P_\ell(x)\, dx + V_2 \int_{-1}^0 P_\ell(x) . dx\right)\left(\frac{2\ell+1}{2a^\ell}\right) = A_\ell$$



$$\left[V_1 . (2\ell+1) \int_0^1 P_\ell(x)\, dx + V_2\, (2\ell+1) . \int_{-1}^0 P_\ell(x) . dx\right] \frac{1}{2a^\ell} = A_\ell$$

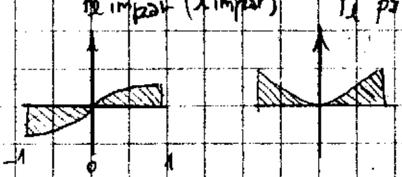
$$\to V_1 . \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$A\ell = \left(\frac{1}{2a^\ell}\right)(2\ell+1)\left[\int_0^1 V_1 . P_\ell\, dx + \int_{-1}^0 V_2 P_\ell\, dx\right]$$

Análisis de los $\int P_\ell$

$$\int_{-1}^0 P_\ell(x) . dx$$

P_ℓ impar (ℓ impar) $\quad$ P_ℓ par (ℓ par)



$$\int_{-1}^0 P_\ell \text{ impar} = -\int_0^1 P_\ell \text{ impar} \qquad \int_{-1}^0 P_\ell \text{ par} = \int_0^1 P_\ell \text{ par}$$

$= \int$

$$\int_{-1}^0 P_\ell = -\int_0^1 P_{(2\ell+1)} + \int_0^1 P_{2\ell}$$

$$B\ell = A\ell . a^{2\ell+1}$$

$$P_\ell(-x) = \left\{\begin{array}{ll} P_\ell(x) & \ell \text{ par} \\ -P_\ell(x) & \ell \text{ impar}\end{array}\right\} = (-1)^\ell P_\ell(x)$$

$$V_2 . (2\ell+1) \int_{-1}^0 P_\ell(x) . dx$$

$x = -x$

$dx = -dx \to$

$$V_2\, (2\ell+1) \int_0^1 P_\ell(-x) . dx$$

$$V_2 . (2\ell+1)\, (-1)^\ell \int_0^1 P_\ell(x)\, dx$$

$$\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}\left[V_1 + V_2\, (-1)^\ell\right]$$

$$\boxed{\phi_I(r,\theta) = \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right)\left[V_1 \int_0^1 P_\ell(x)\, dx + V_2 \int_{-1}^0 P_\ell(x)\, dx\right]\left(\frac{r}{a}\right)^\ell . P_\ell(\cos\theta)}$$

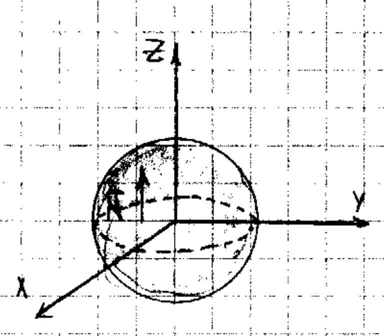
$$\phi_{II}(r,\theta) = \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) \frac{a^{2\ell+1}}{a^\ell}\left[V_1 \int_0^1 P_\ell(x)\, dx + V_2 \int_{-1}^0 P_\ell(x)\, dx\right] r^{-\ell-1} . P_\ell(\cos\theta)$$

$$a^{2\ell} . a = \frac{\cancel{a^\ell} a^\ell . a}{\cancel{a^\ell}}$$

$$\boxed{\phi_{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\left[V_1 \int_0^1 P_\ell(x)\, dx + V_2 \int_{-1}^0 P_\ell(x)\, dx\right]\left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)}$$

5. Imán Esférico

i. $\vec{M} = M_0 \hat{z}$



(a) i) $\dfrac{\delta \vec{m}}{\delta V} = \vec{M} \Rightarrow$

$$\vec{m} = \int_V \vec{M} \, dV$$

$$\vec{m} = \iiint_0^a M_0 \hat{z} \, r^2 \operatorname{sen}\theta \, dr \, d\theta \, d\varphi$$

$$\vec{m} = \frac{1}{3} a^3 M_0 \hat{z} \, 2\pi \, 2$$

$$\boxed{\vec{m} = \frac{4}{3}\pi a^3 M_0 \hat{z}}$$

ii) $\vec{\nabla}\cdot\vec{M} = -\rho_M$ $\qquad \vec{M}\cdot\hat{n} = \sigma_M$

$$\sigma_M = M_0 \hat{z}\cdot\hat{r} = \left(M_0 \cos\theta \, \hat{r} - M_0 \operatorname{sen}\theta \, \hat{\theta}\right)\cdot\hat{r}$$

$$\boxed{\sigma_M = M_0 \cos\theta}$$

Pero en esféricas es:

$\vec{M} = M_0(\cos\theta \hat{r} - \operatorname{sen}\theta \hat{\theta}) \Rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 M_0 \cos\theta\right) + \frac{1}{r \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(-M_0 \operatorname{sen}^2\theta\right) = 0$$

$$M_0 \cos\theta \frac{2r}{r} + (-M_0)\frac{2 \operatorname{sen}\theta \cos\theta}{\operatorname{sen}\theta} = 0$$

$\Rightarrow \quad \vec{\nabla}\cdot\vec{M} = 0 \quad \Rightarrow$

no hay acumulación de cargas magnéticas en volumen $\boxed{\rho_M = 0}$

$$\rho_M = M_0 \cos\theta \, \delta(r-a)$$

$$\vec{m} = \int_V \rho_M \, \vec{x} \, dV$$

$$\vec{m} = \iiint M_0 \cos\theta \, \delta(r-a) \, r^2 \operatorname{sen}\theta \, dr \, d\theta \, d\varphi \, (\#, \#)$$

$$\vec{m} = \hat{z} \, M_0 a^3 \iint \cos^2\theta \operatorname{sen}\theta \, d\theta \, d\varphi$$

$$\vec{m} = M_0 \, 2\pi \, \frac{2}{3} a^3 \hat{z}$$

$$\boxed{\vec{m} = \frac{4}{3}\pi a^3 M_0 \hat{z}}$$

$$\vec{\nabla}\times\vec{M} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & M_0 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{M} = 0 \Rightarrow \boxed{\vec{J}_M = 0}$$

$$\vec{g}_m/c = \vec{M}\times\hat{n} = \left(M_0 \cos\theta \, \hat{r} - M_0 \operatorname{sen}\theta \, \hat{\theta}\right)\times\hat{r}$$

$$\vec{M}\times\hat{n} = M_0 \operatorname{sen}\theta \, \hat{\varphi} \rightarrow$$

$r\theta\varphi r$

$$\boxed{\vec{g}_M = c M_0 \operatorname{sen}\theta \, \hat{\varphi}}$$

$\vec{m} = \dfrac{1}{2c}\displaystyle\int_V \vec{x}\times\vec{J}_m \, dV$ (volumen)

$\vec{m} = \dfrac{I}{2c}\displaystyle\int_\Gamma \vec{x}\times d\vec{\ell}$ (línea) $\qquad \vec{J}_m \, dV$, $J \, ds \, d\vec{\ell}$, $I \, d\vec{\ell}$

usaremos $\int$ en volumen con δ de Dirac $\rightarrow$

$$\vec{m} = \frac{1}{2c}\iiint (r\hat{r})\times\left(c M_0 \operatorname{sen}\theta \, \delta(r-a) \, \hat{\varphi}\right) r^2 \operatorname{sen}\theta \, dr \, d\theta \, d\varphi$$

$$\frac{M_0 c}{2c}\iiint -\hat{\theta}\left[r^3 \operatorname{sen}^2\theta \, \delta(r-a)\right] dr \, d\theta \, d\varphi$$

Las componentes en $\hat{x}, \hat{y}$ son nulas por simetría

$$\vec{m} = \frac{M_0}{2} a^3 \left(+\iint \operatorname{sen}^3\theta \, \hat{z} \, d\theta \, d\varphi\right) + \# \hat{x} + \# \hat{y}$$

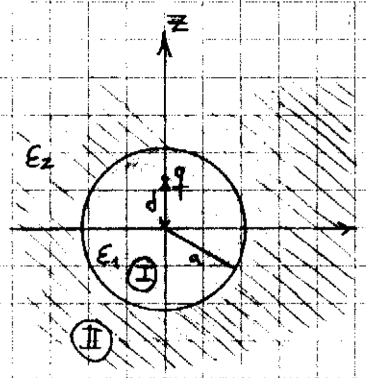
$$\vec{m} = \frac{M_0}{\cancel{2}} a^3 \cdot \cancel{2}\pi \cdot \frac{4}{3} \hat{z} \quad \Rightarrow \quad \boxed{\vec{m} = \frac{4}{3}\pi a^3 M_0 \hat{z}}$$

(b) $\phi_M = \int_V \frac{\rho_m}{|\vec{x}-\vec{x}'|} \cdot dV$ con $\vec{H} = -\vec{\nabla}\phi_M$

$$\phi_M = \int_V M_0 \cos\theta \cdot \delta(r-a) \cdot \left[\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\varphi) \, Y_{lm}^*(\theta',\varphi') \right] dV$$

Como el problema tiene simetría azimutal no puede depender de $\hat{\varphi}$

6.



(a) Así ubicada la carga q (en $\hat{z}$) el problema tiene simetría azimutal.

Podemos pensar en dos zonas donde debemos resolver

I $\quad \nabla^2 \phi = -\frac{4\pi\rho}{\varepsilon_1} = \frac{-4\pi}{\varepsilon_1} \cdot \delta(\vec{r} - d\hat{z}) \cdot q$

II $\quad \nabla^2 \phi = 0$

Nota

II $\varepsilon_2 \vec{\nabla} \cdot \vec{E} = 0 \rightarrow \vec{\nabla} \cdot \vec{E} = 0$

I $\underbrace{\varepsilon_1 \vec{\nabla} \cdot \vec{E}}_{\vec{\nabla} \cdot \vec{D}} = 4\pi\rho$ Es la carga libre q [la única]

También se podría separar en 3 regiones y usar $\nabla^2\phi = 0$ en c/u de ellas.

* simetría azimutal, ~~todo el ángulo~~ $\varphi \rightarrow m = 0$; $E_\varphi = 0$ por simetría de reflexión y rotación en $\varphi \rightarrow \phi \neq \phi(\varphi)$

* usamos $\theta = 0 \rightarrow Q_\ell(\cos\theta)$ no sirve

$$\phi_I(r,\theta) = \frac{q}{\varepsilon_1 |\vec{r} - d\hat{z}|} + \underbrace{\sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1} \right] P_\ell(\cos\theta)}$$

Esta es solución de Laplace

$$\phi_{II}(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1} \right] P_\ell(\cos\theta)$$

$$\phi_{II}(r \to \infty) = 0 \rightarrow A_\ell = 0$$

$$\phi_I(r=0) < +\infty \rightarrow B_\ell = 0$$

$r<d$

$$\phi_I(r,\theta) = \frac{q}{\varepsilon_1} \sum_{\ell=0}^{\infty} \frac{r^\ell}{d^{\ell+1}} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$|\vec{r} - d\hat{z}| = \sqrt{r^2 + d^2 - 2rd\cos\theta}$$

$r>d$

$$\phi_I(r,\theta) = \frac{q}{\varepsilon_1} \sum_{\ell=0}^{\infty} \frac{d^\ell}{r^{\ell+1}} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\phi_{II}(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta)$$

* Continuidad en $r=a$ de ϕ

$$\phi_I(a,\theta) = \phi_{II}(a,\theta)$$

$$\frac{q}{\varepsilon_1} \sum_{\ell=0}^{\infty} \frac{d^\ell}{a^{\ell+1}} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta)$$

$$\frac{1}{\varepsilon_1} q \frac{d^\ell}{a^{\ell+1}} + A_\ell a^\ell = \frac{B_\ell}{a^{\ell+1}}$$

* ~~Salto del campo~~ $\vec{D}$

$$\left(\vec{D}_{II} - \vec{D}_{I}\right)\cdot\hat{n} = 4\pi\sigma_L = 0$$

$$D_I\hat{n} = D_{II}\hat{n} \rightarrow$$

$$\varepsilon_1 \frac{\partial \phi_I}{\partial r}\Big|_{r=a} = \varepsilon_2 \frac{\partial \phi_{II}}{\partial r}\Big|_{r=a}$$

$$\varepsilon_1 \cdot \sum_{l=0}^{\infty} \frac{\partial}{\partial r}\left(\left[\frac{q}{\varepsilon_1}\frac{d^l}{r^{l+1}} + A_l\, r^l\right] P_l(\cos\theta)\right) = \varepsilon_2 \sum_{l=0}^{\infty} \frac{\partial}{\partial r}\left(\frac{B_l}{r^{l+1}}\right) \cdot P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty} \cancel{\varepsilon_1}\left[\frac{q}{\cancel{\varepsilon_1}}\cdot\frac{d^l(-l-1)}{r^{l+2}} + A_l\, l\, r^{l-1}\cdot\varepsilon_1\right] P_l(\cos\theta)\Big|_{r=a} = \sum_{l=0}^{\infty} \frac{B_l(-l-1)}{r^{l+2}}\varepsilon_2 \cdot P_l(\cos\theta)\Big|_{r=a}$$

$$\sum_{l=0}^{\infty}\left(-q\frac{d^l(l+1)}{a^{l+2}} + A_l\, l\, a^{l-1}\varepsilon_1\right) P_l(\cos\theta) = \sum_{l=0}^{\infty} -\frac{B_l(l+1)}{a^{l+2}}\varepsilon_2 \cdot P_l(\cos\theta)$$

$$-q\frac{d^l(l+1)}{a^{l+2}} + A_l\, l\, a^{l-1}\varepsilon_1 = -\frac{B_l(l+1)\varepsilon_2}{a^{l+2}}$$

Usando la ecuación de la continuidad $\rightarrow$

$$\frac{q}{\varepsilon_1}\cdot\frac{d^l}{a^{l+1}} + A_l\cdot a^l = \frac{B_l}{a^{l+1}}$$

$$-\frac{q\, d^l(l+1)}{a^{l+2}} + A_l\, l\, a^{l-1}\varepsilon_1 + \left[\frac{q\, d^l}{\varepsilon_1} + A_l\, a^{2l+1}\right]\frac{(l+1)\varepsilon_2}{a^{l+2}} = 0$$

$$-\frac{q\, d^l(l+1)}{a^{l+2}} + A_l\, l\, a^{l-1}\varepsilon_1 + \frac{q\, d^l}{\varepsilon_1}\frac{\varepsilon_2(l+1)}{a^{l+2}} + A_l(l+1)\varepsilon_2 a^{l-1} = 0$$

$$A_l\left[l\, a^{l-1}\varepsilon_1 + (l+1)\, a^{l-1}\varepsilon_2\right] = \frac{q\, d^l(l+1)}{a^{l+2}}\left[1 - \frac{\varepsilon_2}{\varepsilon_1}\right]$$

$$A_l = \frac{q\, d^l(l+1)\left[1 - \varepsilon_2/\varepsilon_1\right] a^{-l-2}}{a^{l-1}\left[l\,\varepsilon_1 + l\,\varepsilon_2 + \varepsilon_2\right]}$$

$$\boxed{A_l = \frac{q\, d^l(l+1)\left[\varepsilon_1 - \varepsilon_2\right]}{\varepsilon_1\, a^{2l+1}\left[\varepsilon_1 l + \varepsilon_2(l+1)\right]}}$$

$$\rightarrow A_0 = \frac{q\,(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 a\,\varepsilon_2}$$

$$B_l = \frac{q\, d^l}{\varepsilon_1} + \cancel{a^{2l+1}}\frac{q\, d^l(l+1)\left[\varepsilon_1 - \varepsilon_2\right]}{\varepsilon_1\cancel{a^{2l+1}}\left[\varepsilon_1 l + \varepsilon_2(l+1)\right]}$$

$$B_l = \frac{q\, d^l}{\varepsilon_1}\left[1 + \frac{(l+1)\left[\varepsilon_1 - \varepsilon_2\right]}{\left[\varepsilon_1 l + \varepsilon_2(l+1)\right]}\right]$$

$$\left[\frac{\varepsilon_1 l + \cancel{\varepsilon_2(l+1)} + (l+1)\varepsilon_1 - \cancel{\varepsilon_2(l+1)}}{\left[\varepsilon_1 l + \varepsilon_2(l+1)\right]}\right]$$

$$B_l = \frac{q\, d^l}{\cancel{\varepsilon_1}}\cdot\frac{\cancel{\varepsilon_1}(2l+1)}{\varepsilon_1 l + \varepsilon_2(l+1)}$$

$$\boxed{B_l = \frac{q\, d^l(2l+1)}{\varepsilon_1 l + \varepsilon_2(l+1)}} \rightarrow B_0 = \frac{q}{\varepsilon_2}$$

$$\boxed{\begin{aligned}
\phi_{I}(r,\theta)_{r<d} &= \sum_{l=0}^{\infty}\left(\frac{q}{\varepsilon_1}\frac{r^l}{d^{l+1}} + \frac{r^l\, q\, d^l (l+1)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1\, a^{2l+1}[\varepsilon_1 l + \varepsilon_2(l+1)]}\right) P_l(\cos\theta)\\
\phi_{I}(r,\theta)_{r>d} &= \sum_{l=0}^{\infty}\left(\frac{q}{\varepsilon_1}\frac{d^l}{r^{l+1}} + \frac{r^l\, q\, d^l (l+1)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1\, a^{2l+1}[\varepsilon_1 l + \varepsilon_2(l+1)]}\right) P_l(\cos\theta)\\
\phi_{II}(r,\theta) &= \sum_{l=0}^{\infty}\left(\frac{q\, d^l (2l+1)}{\varepsilon_1 l + \varepsilon_2 (l+1)}\right)\frac{1}{r^{l+1}} P_l(\cos\theta)
\end{aligned}}$$

(b)

$$\sigma_p = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}$$
$$\rho_p = -\vec{\nabla}\cdot\vec{P}$$

$$\vec{P} = \chi\vec{E}$$

$$\varepsilon = 1 + 4\pi\chi$$

$$\frac{\varepsilon - 1}{4\pi} = \chi$$

$$\vec{P} = \left(\frac{\varepsilon-1}{4\pi}\right)\vec{E}$$

$$-\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{\varepsilon-1}{4\pi}\right)\vec{E}$$

$$\rho_p = -\left(\frac{\varepsilon-1}{4\pi}\right)\vec{\nabla}\cdot\vec{E}$$

$$\rho_p = -\left(\frac{\varepsilon_1-1}{4\pi}\right)\frac{4\pi}{\varepsilon_1}\, q\,\delta(\vec{r}-d\hat{z})$$

Región I $\boxed{\rho_p = \left(\frac{1-\varepsilon_1}{\varepsilon_1}\right) q\,\delta(\vec{r}-d\hat{z})}$

Región II $\boxed{\rho_p = 0}$ pues $\vec{\nabla}\cdot\vec{E} = 0$ y ε_2 es escalar constante

La $\hat{n}$ para este caso es $\hat{r}$ con lo cual basta con calcular $-\frac{\partial\phi}{\partial r} = E_{\hat{n}}$

$$P_{II}\hat{n} = \left(\frac{\varepsilon_1-1}{4\pi}\right) E_{II}\hat{n}$$

$$P_{I}\hat{n} = \left(\frac{\varepsilon_2-1}{4\pi}\right) E_{I}\hat{n}$$

CASO LÍMITE

Si $d=0 \Rightarrow$ se deben matar los términos de la serie $l \geq 1$ y resulta

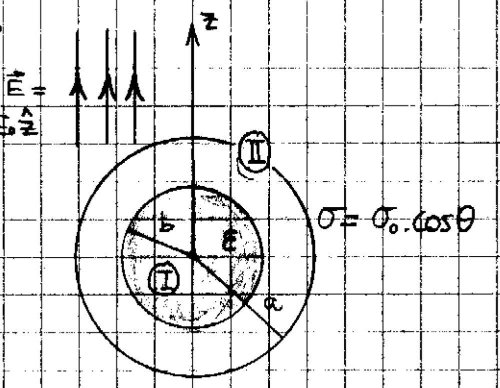
$$\phi_I(r,\theta) = \frac{q}{\varepsilon_1 r} + \frac{q\,(\varepsilon_1-\varepsilon_2)}{\varepsilon_1\, a\, \varepsilon_2}$$

$$\phi_{II}(r,\theta) = \frac{q}{\varepsilon_2 r}$$

$$\sigma_p = -\left[\left(\frac{\varepsilon_1-1}{4\pi}\right)E_{II}\hat{n} - \left(\frac{\varepsilon_2-1}{4\pi}\right)E_{I}\hat{n}\right]$$

$$\boxed{\begin{aligned}\sigma_p = -\Bigg[&\sum_{l=0}^{\infty} -\left(\frac{\varepsilon_1-1}{4\pi}\right)\frac{q\, d^l (2l+1)}{[\varepsilon_1 l + \varepsilon_2(l+1)]}\frac{(l+1)}{r^{l+2}} P_l(\cos\theta)\\
&- \sum_{l=0}^{\infty}\left(\frac{\varepsilon_2-1}{4\pi}\right)\left[-\frac{q\, d^l (l+1)}{\varepsilon_1\, r^{l+2}} + \frac{l\, r^{l-1}\, q\, d^l (l+1)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1\, a^{2l+1}[\varepsilon_1 l + \varepsilon_2(l+1)]}\right] P_l(\cos\theta)\Bigg]_{r=a}\end{aligned}}$$

7.



(a) Como la distribución de carga de la esfera es función de θ → para θ fijo a cualquier φ se ve igual ⇒ el problema tiene simetría de rotación en φ y reflexión en planos XZ, YZ con lo cual $E_\varphi = 0$ → $m = 0$ si usamos esféricas (empleo $0<\varphi<2\pi$)

Puedo dividir en 3 zonas y resolver Laplace en c/u pero opto por 2 zonas

(I) $0<r<b$ $\quad \nabla^2\phi = 0$ Laplace

(II) $b<r$ $\quad \nabla^2\phi = -4\pi\rho$ Poisson

* Región (I)

La bola tiene permitividad ε (LIH)

Usamos $\theta=0$, $\theta=\pi$ → $Q_\ell(\cos\theta)$ no sirve

$$\phi_I(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \underbrace{B_\ell}_{\to 0} r^{-\ell-1}\right] P_\ell(\cos\theta)$$

→0 pues $\phi_I(r=0) < \infty$

$$\phi_I(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

* Región (II)

En la región II resolvemos la integral de Poisson más la de Laplace

$$\rho(\vec{r}) = \sigma_0 \cos\theta\, \delta(r-a)$$

$$\phi_{cáscara}(r,\theta) = \int_V \frac{\rho\, dV}{|\vec{r}-\vec{r}'|} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\sigma_0 \cos\theta'\, \delta(r'-a)\, r'^2 \operatorname{sen}\theta'\, dr'\, d\theta'\, d\varphi'}{|\vec{r}-\vec{r}'|}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\sigma_0 \cos\theta'\, a^2 \operatorname{sen}\theta'\, d\theta'\, d\varphi'}{|\vec{r} - a\hat{r}'|}$$

$$\phi_{cáscara}(r,\theta) = \int_0^{2\pi}\int_0^{\pi} \sigma_0 \cos\theta'\, a^2 \operatorname{sen}\theta' \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)\, d\theta'\, d\varphi'$$

$r>a \longrightarrow \dfrac{a^\ell}{r^{\ell+1}}$

$r<a \longrightarrow \dfrac{r^\ell}{a^{\ell+1}}$

ojo $|\vec{r}-\vec{r}'|$ depende de φ pero en la integración tal dependencia debe desaparecer porque $\phi \neq \phi(\varphi)$ por la simetría

Pero $m=0$ →

$$\phi_{cáscara}(r,\theta) = \int_0^{2\pi}\int_0^{\pi} \sigma_0 \cos\theta' \operatorname{sen}\theta'\, a^2 \left[\sum_{\ell=0}^{\infty} \frac{4\pi}{(2\ell+1)} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell 0}^*(\theta',\varphi) Y_{\ell 0}(\theta,\varphi)\right] d\theta'\, d\varphi'$$

$$\phi_{cáscara}(r,\theta) = \int_0^{2\pi}\int_0^{\pi} \sigma_0 a^2 \cos\theta' \operatorname{sen}\theta' \left[\sum_{\ell=0}^{\infty} \frac{\cancel{4\pi}}{\cancel{(2\ell+1)}} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{\cancel{(2\ell+1)}}{\cancel{4\pi}} P_\ell(\cos\theta') P_\ell(\cos\theta)\right] d\theta'\, d\varphi'$$

$$\phi_{Laplace}(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

El campo uniforme lo veo como un potencial $-E_0 z = -E_0 r\cos\theta$ que es condición de contorno $\phi(r\to\infty) \longrightarrow -E_0 r \cos\theta$

Luego $\phi_{cascara}$ no aporta en $r\to+\infty$ y para Laplace necesitaré:

$$\sum_{l=0}^{\infty}\left(A_l r^l + \underbrace{\frac{B_l}{r^{l+1}}}_{\to 0}\right)P_l(\cos\theta) = -E_0 . r . \cos\theta \Rightarrow \begin{matrix} A_l = 0 & l>1 \\ A_1 \neq 0 & \end{matrix} \quad A_1 r . \cos\theta = -E_0 r \cos\theta \Rightarrow A_1 = -E_0$$

$$\phi_{II}(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{\sigma_0 a^2}{} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) . 2\pi\int_0^{\pi}\cos\theta' \operatorname{sen}\theta' . d\theta' \; P_l(\cos\theta')\right)$$

$$+\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} P_l(\cos\theta) - E_0 . r . \cos\theta$$

$r<a$

$$\sum_{l=0}^{\infty}\frac{\sigma_0 2\pi r^l}{a^{l-1}} . P_l(\cos\theta) . \mathrm{I}$$

* Continuidad en $r=b$

$$\phi_I(b,\theta) = \phi_{II}(b,\theta)$$

$$\sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta) = \sum_{l=0}^{\infty}\sigma_0 a^2 . \frac{b^l}{a^{l+1}} . P_l(\cos\theta) . 2\pi\int_0^{\pi}\cos\theta' \operatorname{sen}\theta' . P_l(\cos\theta') . d\theta'$$

$$+\frac{B_l}{b^{l+1}} P_l(\cos\theta) - E_0 . b . \cos\theta$$

* Ortogonalidad

$$\sum_{l=0}^{\infty} A_l b^l \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) . \operatorname{sen}\theta . d\theta = \sum_{l=0}^{\infty}\frac{\sigma_0}{2}\frac{b}{a^{l-1}}\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \operatorname{sen}\theta . d\theta$$

$$\int_0^{\pi}\cos\theta' . \operatorname{sen}\theta' P_l(\cos\theta') . d\theta' + \frac{B_l}{b^{l+1}}\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) . \operatorname{sen}\theta . d\theta$$

$$-E_0 . b \int_0^{\pi}\cos\theta . P_{l'}(\cos\theta) . \operatorname{sen}\theta . d\theta$$

$$A_l b^l . \frac{2}{(2l+1)} = \frac{\sigma_0 b}{2 a^{l-1}}\left(\frac{2}{2l+1}\right)\int_0^{\pi}\cos\theta' . \operatorname{sen}\theta' . P_l(\cos\theta') . d\theta'$$

$$+\frac{B_l}{b^{l+1}}\left(\frac{2}{2l+1}\right) - E_0 . b . \int_0^{\pi}\cos\theta . \operatorname{sen}\theta . P_l(\cos\theta) . d\theta$$

$$\boxed{\int_0^{\pi}\cos\theta \operatorname{sen}\theta . P_l(\cos\theta) . d\theta = -\int_1^{-1} x . P_l(x) . dx = +\int_{-1}^{+1} x . P_l(x) . dx}$$

$\cos\theta = x$
$-\operatorname{sen}\theta . d\theta = dx$

$$A_l = \frac{\sigma_0}{b^{l-1} a^{l-1} . 2}\int_{-1}^{+1} x . P_l(x) . dx + \frac{B_l}{b^{2l+1}} - \frac{(2l+1)}{2}\frac{E_0}{b^{l-1}}\int_{-1}^{+1} x . P_l(x) . dx$$

Esto conduce a un atolladero; probaremos boundary conditions

* Salto del Campo $\vec{D}$

$$\vec{E}^{\hat{n}}_{II}\Big|_{r=b} - \varepsilon.\vec{E}^{\hat{n}}_{I}\Big|_{r=b} = 4\pi\sigma_L = 0 \qquad -\frac{\partial\phi_{II}}{\partial r}\Big|_{r=b} + \varepsilon.\frac{\partial\phi_I}{\partial r}\Big|_{r=b} = 0$$

$$\sum_{l=1}^{\infty} \sigma_0 2\pi.l\frac{b^{l-1}}{a^{l-1}}.P_l(\cos\theta).I - \frac{(l+1).B_l}{b^{l+2}}P_l(\cos\theta) - E_0\cos\theta$$

$$= \sum_{l=1}^{\infty} \varepsilon.A_l.l\,b^{l-1}.P_l(\cos\theta)$$

$$l=1 \rightarrow \quad \sigma_0 2\pi.I - \frac{2B_1}{b^3} - E_0\cos\theta = 0 \rightarrow \boxed{B_1 = \frac{b^3\sigma_0 I}{4} - \frac{E_0\cos\theta\, b^3}{2}} \quad ①$$

$$l>1 \quad \sum_{l=1}^{\infty} 2\pi\sigma_0\frac{l\,b^{l-1}}{a^{l-1}}.I.P_l(\cos\theta) - \frac{(l+1)B_l}{b^{l+2}}P_l(\cos\theta) - \varepsilon A_l l\, b^{l-1}.P_l(\cos\theta) = 0$$

$$\frac{2\pi\sigma_0 l\, b^{l-1}}{a^{l-1}}I - \frac{(l+1)B_l}{b^{l+2}} - \varepsilon.A_l.l.b^{l-1} = 0$$

$$B_l = \left(-\varepsilon A_l.l\;b^{l-1} + \frac{\sigma_0\,l\,b^{l-1}}{2.a^{l-1}}.I\right)\frac{b^{l+2}}{(l+1)}$$

$$\boxed{B_l = \frac{-\varepsilon A_l\,l\,b^{2l+1}}{l+1} + \frac{2\pi\sigma_0\,l.b^{2l+1}.I}{a^{l-1}(l+1)}} \quad ②$$

* Continuidad de $E_{\hat{t}}$

$$-\frac{1}{r}\frac{\partial\phi_I}{\partial\theta}\Big|_{r=b} = -\frac{1}{r}\frac{\partial\phi_{II}}{\partial\theta}\Big|_{r=b} \Rightarrow \frac{\partial\phi_I}{\partial\theta}\Big|_{r=b} = \frac{\partial\phi_{II}}{\partial\theta}\Big|_{r=b}$$

$$\sum_{l=1}^{\infty} 2\pi\sigma_0\frac{b^l}{a^{l-1}}.I\frac{\partial}{\partial\theta}\left[P_l(\cos\theta)\right] + \frac{B_l}{b^{l+1}}\frac{\partial}{\partial\theta}\left[P_l(\cos\theta)\right] + E_0.b.\mathrm{sen}\,\theta = \sum_{l=0} A_l\,b^l\frac{\partial}{\partial\theta}\left[P_l(\cos\theta)\right]$$

$$l=1 \quad -2\pi\sigma_0 b\,I\,\mathrm{sen}\theta - \frac{B_1}{b^2}.\mathrm{sen}\theta + E_0 b\,\mathrm{sen}\theta = -A_1 b\,\mathrm{sen}\theta$$

$$\boxed{B_1 = -2\pi\sigma_0 b\,I + E_0 b + A_1 b} \quad ③$$

$$l>1 \quad \frac{\sigma_0 b^l}{2.a^{l-1}}I + \frac{B_l}{b^{l+1}} - A_l b^l = 0$$

$$B_l = \left(A_l b^l - \frac{2\pi\sigma_0 b^l I}{a^{l-1}}\right)b^{l+1}$$

$$\boxed{B_l = A_l.b^{2l+1} - \frac{2\pi\sigma_0.b^{2l+1}.I}{a^{l-1}}} \quad ④$$

Deberían cumplirse a la vez las ecuaciones ② y ④ $\Rightarrow$

$$\left(-\frac{\varepsilon A_l l}{l+1} + \frac{2\pi\sigma_0 l\,I}{a^{l-1}(l+1)}\right)\cancel{b^{2l+1}} = \left(A_l - \frac{2\pi\sigma_0 I}{a^{l-1}}\right)\cancel{b^{2l+1}}$$

$$\frac{2\pi\sigma_0 I}{a^{l-1}}\left(\frac{l}{l+1} + 1\right) = A_l\left(1 + \frac{\varepsilon l}{l+1}\right)$$

$$\frac{2\pi\sigma_0 I}{a^{l-1}}\frac{(l+l)}{(l+1)} = A_l\frac{(l+1+\varepsilon l)}{(l+1)}$$

$$\frac{2\pi\sigma_0 I}{a^{\ell-1}}(\ell+2) = A_\ell\left(\ell(\varepsilon+1)+1\right) \rightarrow \boxed{A_\ell = \frac{2\pi\sigma_0 I(\ell+2)}{a^{\ell-1}\left(\ell[\varepsilon+1]+1\right)}}$$

$$B_\ell = \frac{2\pi\sigma_0 I(\ell+2)\, b^{2\ell+1}}{a^{\ell-1}\left(\ell[\varepsilon+1]+1\right)} - \frac{\sigma_0 b^{2\ell+1} I}{2a^{\ell+1}}$$

$$\frac{2\pi\sigma_0 I\, b^{2\ell+1}}{a^{\ell-1}}\left[\frac{\cancel{\ell}+2-1-\ell\varepsilon-\cancel{\ell}}{\ell[\varepsilon+1]+1}\right]$$

$$\boxed{B_\ell = \frac{2\pi\sigma_0 I\, b^{2\ell+1}}{a^{\ell-1}}\left(\frac{1-\ell\varepsilon}{1+\ell[\varepsilon+1]}\right)}$$

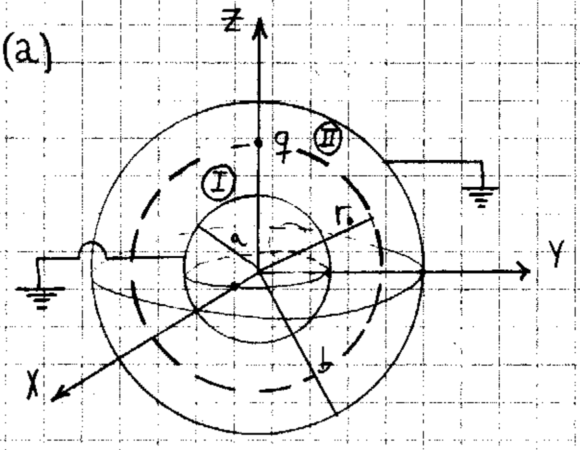
$$\phi_{\text{cáscara}}(r,\theta) = \int_0^{2\pi}\int_0^{\pi}\sigma_0 a^2 \cos\theta' \cdot \operatorname{sen}\theta' \left[\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta') \cdot P_\ell(\cos\theta)\right] d\theta' \cdot d\varphi'$$

por completitud $\left\{\right.$

$$\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}} \cdot \delta(\cos\theta - \cos\theta')$$

8.

(a)



Ubicamos la carga en el eje z para tener preciosa simetría azimutal
simetría rotación en φ
" reflexión en XZ, ZY

$$E_\varphi = 0 \rightarrow \phi \neq \phi(\varphi)$$

De $r>b$ y $r<a$ no hay que preocuparse porque al estar las esferas a tierra aislan de campo y entonces:

$$\phi(r>b) = 0 \quad , \quad \phi(r<a) = 0$$

Separamos con una esfera de radio r_0 en dos regiones (I), (II) donde vale $\nabla^2\phi =$

$$\rho(\vec{r}) = \frac{q}{r^2 \operatorname{sen}\theta} \cdot \delta(r-r_0) \cdot \delta(\theta-\theta') \cdot \delta(\varphi-\varphi')$$

$$\phi^{I}_{\ell m} = \left[A_\ell \cdot r^\ell + B_\ell r^{-\ell-1}\right] \cdot \left[C_\ell P^m_\ell + D_\ell \cdot Q^m_\ell\right] \cdot \left[E_m \cdot \cos(m\varphi) + F_m \cdot \operatorname{sen}(m\varphi)\right]$$

Como $\phi \neq \phi(\varphi) \Rightarrow Q(\varphi) = cte \Rightarrow m = 0$ y reabsorbemos la constante

Como $\theta = 0$, $\theta = \pi$ están en la zona de interés $\Rightarrow Q^m_\ell$ no sirve $\Rightarrow D_\ell = 0$

$$\phi^{I}_{\ell 0}(r,\theta) = \sum_{\ell=0}^{\infty} \left(A^I_\ell \cdot r^\ell + B^I_\ell \cdot r^{-\ell-1}\right) \cdot P_\ell(\cos\theta)$$

* Región (I)

$$\phi^{I}_{\ell 0}(a,\theta) = 0 \rightarrow R(r=a) = 0 \rightarrow 0 = A^I_\ell \cdot a^\ell + B^I_\ell \cdot a^{-\ell-1}$$

$$B^I_\ell = -\frac{A^I_\ell \cdot a^\ell}{a^{-\ell-1}}$$

$$B^I_\ell = -A^I_\ell \cdot a^{2\ell+1}$$

$$\phi^{I}_{\ell 0}(r,\theta) = \sum_{\ell=0}^{\infty} A^I_\ell \left[r^\ell - a^{2\ell+1} \cdot r^{-\ell-1}\right] \cdot P_\ell(\cos\theta)$$

* Región (II)

$$\phi^{II}_{\ell 0}(b,\theta) = 0 \rightarrow R(r=b) = 0 \rightarrow 0 = A^{II}_\ell \cdot b^\ell + B^{II}_\ell \cdot b^{-\ell-1}$$

$$B^{II}_\ell = -A^{II}_\ell \cdot b^{2\ell+1}$$

$$\phi^{II}_{\ell 0}(r,\theta) = \sum_{\ell=0}^{\infty} A^{II}_\ell \left[r^\ell - b^{2\ell+1} \cdot r^{-\ell-1}\right] \cdot P_\ell(\cos\theta)$$

* Continuidad del potencial en $r=r_0$

$$\phi_{l_0}^{I}(r_0,\theta) = \phi_{l_0}^{II}(r_0,\theta)$$

$$\sum_{l=0}^{\infty} A_l^{I}\left(r_0^{l} - a^{2l+1}\cdot r_0^{-l-1}\right) P_l(\cos\theta) = \sum_{l=1}^{\infty} A_l^{II}\left(r_0^{l} - b^{2l+1}\cdot r_0^{-l-1}\right) P_l(\cos\theta)$$

$$A_l^{I}\left[r_0^{l} - a^{2l+1}\cdot r_0^{-l-1}\right] = A_l^{II}\left[r_0^{l} - b^{2l+1}\cdot r_0^{-l-1}\right]$$

$$A_l^{I}\frac{\left(r_0^{l} - a^{2l+1}\, r_0^{-l-1}\right)}{\left(r_0^{l} - b^{2l+1}\, r_0^{-l-1}\right)} = A_l^{II}$$

defino $A_l^{I} = A_l \rightarrow$

$$\phi_{l_0}^{I}(r,\theta) = \sum_{l=0}^{\infty} A_l\left[r^{l} - a^{2l+1}\, r^{-l-1}\right]. P_l(\cos\theta)$$

$$\phi_{l_0}^{II}(r,\theta) = \sum_{l=0}^{\infty} A_l\cdot\left(\frac{r_0^{l} - a^{2l+1}\, r_0^{-l-1}}{r_0^{l} - b^{2l+1}\, r_0^{-l-1}}\right)\cdot\left[r^{l} - b^{2l+1}\, r^{-l-1}\right]. P_l(\cos\theta)$$

* Salto del campo en $r=r_0$

$$\rho = \frac{q}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos 0)\,\delta(\varphi-\varphi')$$

$$\rho = \frac{q}{r^2 \operatorname{sen}\theta}\,\delta(r-r_0)\,\delta(\theta-0)\,\delta(\varphi)$$

$$\sigma = \frac{q}{r_0^2}\,\delta(\theta)$$

$$\left[\frac{\partial\phi_I}{\partial r} - \frac{\partial\phi_{II}}{\partial r}\right]_{r=r_0} = 4\pi\sigma$$

$$\sum_{l=}^{\infty} A_l\left[l\, r_0^{l-1} - a^{2l+1}(-l-1)\, r_0^{-l-2}\right] P_l(\cos\theta) -$$

$$\sum_{l=0}^{\infty} A_l\left(\frac{r_0^{l} - a^{2l+1}\, r_0^{-l-1}}{r_0^{l} - b^{2l+1}\, r_0^{-l-1}}\right)\left[l\, r_0^{l-1} - b^{2l+1}\,(-l-1)\, r_0^{-l-2}\right]. P_l(\cos\theta) = 4\pi\,\frac{q}{r_0^2 \operatorname{sen}\theta}\,\delta(\theta)$$

$$\int_0^{\pi}\sum_{l=0}^{\infty} A_l\left[l\, r_0^{l-1} - a^{2l+1}(-l-1)\, r_0^{-l-2} - \left[\frac{r_0^{l} - a^{2l+1}\, r_0^{-l-1}}{r_0^{l} - b^{2l+1}\, r_0^{-l-1}}\right] l\, r_0^{l-1} - b^{2l+1}(-l-1)\, r_0^{-l-2}\right] P_l(\cos\theta)\, P_{l'}(\cos\theta)\, \operatorname{sen}\theta$$

$$= \int_0^{\pi} 4\pi\, q\cdot\frac{1}{r_0^2}\,\delta(\theta)\, P_{l'}(\cos\theta)\,\frac{\cancel{\operatorname{sen}\theta}}{\cancel{\operatorname{sen}\theta}}\, d\theta$$

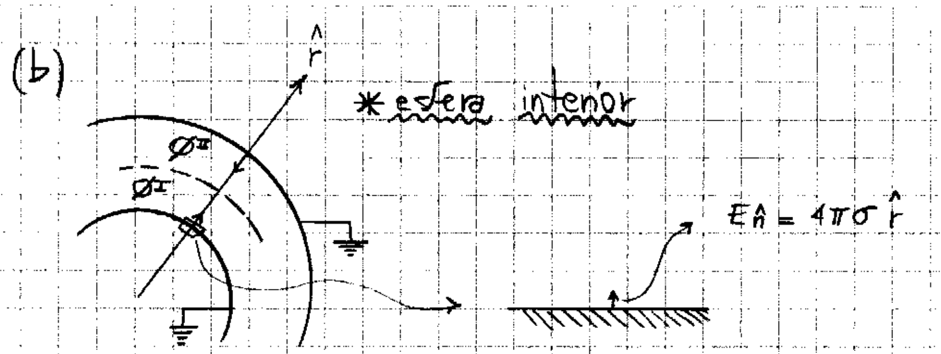
$$\underbrace{A_l\left[l\, r_0^{l-1} + a^{2l+1}(l+1)\, r_0^{-l-2} - \left[\frac{r_0^{l} - a^{2l+1}\, r_0^{-l-1}}{r_0^{l} - b^{2l+1}\, r_0^{-l-1}}\right] l\, r_0^{l-1} + b^{2l+1}(l+1)\, r_0^{-l-2}\right]}_{= f(r_0,a,b)}\frac{2}{2l+1} = 4\pi q\,\frac{1}{r_0^2}\,\overbrace{P_l(1)}^{1\ \forall l}$$

$$A_l = \frac{(2l+1)\, 2\pi\, q}{r_0^2\,\left(f(r_0,a,b)\right)}$$

$$\boxed{\phi_{l_0}^{I}(r,\theta) = \sum_{l=0}^{\infty}\frac{2\pi}{r_0^2}\, q\,\frac{(2l+1)}{f(a,b,r_0)}\cdot\left[r^{l} - a^{2l+1}\, r^{-l-1}\right] P_l(\cos\theta)}$$

$$\boxed{\phi_{l_0}^{II}(r,\theta) = \sum_{l=0}^{\infty}\frac{2\pi}{r_0^2}\, q\,\frac{(2l+1)}{f(a,b,r_0)},\left(\frac{r_0^{l} - a^{2l+1}\, r_0^{-l-1}}{r_0^{l} - b^{2l+1}\, r_0^{-l-1}}\right)\left[r^{l} - b^{2l+1}\, r^{-l-1}\right] P_l(\cos\theta)}$$

(b)



$$\left.\frac{\partial \phi^I}{\partial r}\right|_{r=a} = 4\pi\sigma$$

$$\sum_{l=0}^{\infty} \frac{2\pi}{r_0^2} q \cdot \frac{(2l+1)}{f(r_0,a,b)} \left[l\, r^{l-1} - a^{2l+1} \cdot (-l-1) \cdot r^{-l-2} \right]_{r=a} P_l(\cos\theta) = 4\pi\sigma$$

$$\left(l \cdot a^{l-1} + a^{2l+1} (l+1)\, a^{-l-2} \right)$$

$$\left[\frac{l \cdot a^l}{a} + \frac{\cancel{a^l}\, a^l\, \cancel{a}}{\cancel{a^l}\, a^{2}} (l+1) \right]$$

$$\frac{l \cdot a^l}{a} + \frac{l \cdot a^l}{a} + \frac{a^l}{a}$$

densidad de carga sobre esfera $R=a$

$$\boxed{\sum_{l=0}^{\infty} \frac{q(2l+1)}{2r_0^2} \frac{(2l\, a^{l-1} + a^{l-1})}{f(r_0,a,b)} \cdot P_l(\cos\theta) = \sigma(\theta)}$$

$$Q_{IND}^{TOT} = \int_0^{2\pi}\int_0^{\pi} \sigma(\theta) \cdot a^2 \operatorname{sen}\theta \cdot d\theta\, d\varphi = \sum_{l=1}^{\infty} \frac{q}{2r_0^2} \overset{(2l+1)}{\downarrow} \left(\frac{2l a^{l-1} + a^{l-1}}{f(r_0,a,b)} \right) \int_0^{2\pi}\int_0^{\pi} P_l(\cos\theta) \cdot \operatorname{sen}\theta \cdot a^2 d\theta \cdot d\varphi$$

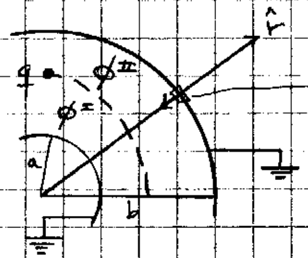
$$(2l+1) \int_0^1 P_l(x) \cdot dx = \left(\frac{1}{2}\right)^{\frac{l-1}{2}} \cdot \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$

$$(2l+1) \int_0^{\pi/2} P_l(\cos\theta) \cdot \operatorname{sen}\theta \cdot d\theta$$

$\cos\theta = x$
$-\operatorname{sen}\theta . d\theta = dx$
$\cos\theta = 1$
θ: $\cos$

$$\boxed{Q_{IND}^{TOT} = \sum_{l=0}^{\infty} \frac{q}{2r_0^2} \frac{[2l a^{l-1} + a^{l-1}]}{f[r_0,a,b]} \cdot 2\pi \cdot a^2 \cdot \left(-\frac{1}{2}\right)^{\frac{l-1}{2}} \cdot \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}}$$

* esfera exterior



$$-\frac{\partial \phi^{II}}{\partial r}\bigg|_{r=b} = 4\pi\sigma$$

$$-\sum_{l=0}^{\infty}\frac{(2l+1)}{r_0^2}2\pi q\,\frac{1}{f(r_0,a,b)}\left[\frac{r_0^l - a^{2l+1}\cdot r_0^{-l-1}}{r_0^l - b^{2l+1}\cdot r_0^{-l-1}}\right]\left[l\,b^{l-1} - b^{2l+1}(-l-1)\,b^{-l-2}\right]P_l(\cos\theta)$$

$$= 4\pi\sigma(\theta)$$

$$\left[l\,b^{l-1} + b^{2l+1}\,l\,b^{-l-2} + b^{2l+1}\,b^{-l-2}\right]$$

$$(2l\,b^{l-1} + b^{l-1})$$

Densidad de carga sobre la esfera con $r=b$

$$\boxed{\sigma(\theta) = \sum_{l=0}^{\infty}\frac{(2l+1)}{r_0^2}\frac{q}{2}\cdot\frac{1}{f(r_0,a,b)}\left[\frac{r_0^l - a^{2l+1}\cdot r_0^{-l-1}}{r_0^l - b^{2l+1}\cdot r_0^{-l-1}}\right](2l\,b^{l-1} + b^{l-1})\cdot P_l(\cos\theta)}$$

$$Q_{IND}^{TOT} = \int_0^{2\pi}\int_0^{\pi}\sigma(\theta)\,b^2\,\mathrm{sen}\theta\,d\theta\,d\varphi$$

$$Q_{ind}^{TOT} = \sum_{l=1}^{\infty}\frac{(2l+1)}{r_0^2}\cdot\frac{q}{2}\cdot\frac{1}{f(r_0,a,b)}\left[\qquad\right](\qquad)\,b^2\,2\pi\int_0^{\pi}P_l(\cos\theta)\,\mathrm{sen}\theta\,d\theta$$

$$\boxed{Q_{IND}^{TOT} = \sum_{l=0}^{\infty}\frac{(2l+1)}{r_0^2}\frac{q}{2}\frac{1}{f(r_0,a,b)}\left[\frac{r_0^l - a^{2l+1}\cdot r_0^{-l-1}}{r_0^l - b^{2l+1}\cdot r_0^{-l-1}}\right](2l\,b^{l-1} + b^{l-1})\cdot\frac{2\pi b^2}{(2l+1)}\left(-\frac{1}{2}\right)^{\frac{l-1}{2}}\frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}}$$

(c)

Sea $b\to\infty$ se presentan inconvenientes en el término

$\lim b\to\infty$

$$\frac{1}{f(r_0,a,b)}\to 0$$

pues

$$f(r_0,a,b)\to\infty$$

$\lim b\to\infty$

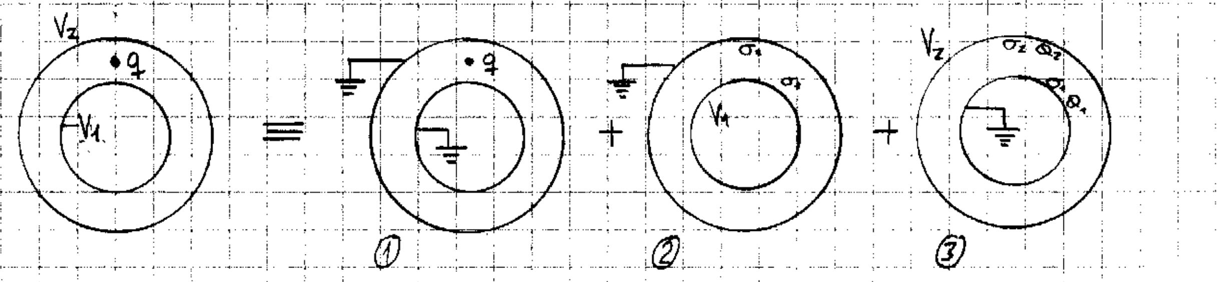
$$\left(\frac{2l\,b^{l+1} + b^{l+1}}{r_0^l - b^{2l+1}\cdot r_0^{-l-1}}\right)\longrightarrow\frac{\infty}{\infty}$$

$$\lim_{b\to\infty}\frac{\left(\frac{2l}{b^l}\right)^{\to 0} + \left(\frac{1}{b^l}\right)^{\to 0}}{\underbrace{\frac{r_0^l}{b^{2l+1}}}_{\to 0} - \underbrace{r_0^{-l-1}}_{\not\to 0}} = 0$$

$$\frac{b^l b}{b^l b^l b}$$

En el caso en que $b\to\infty$ $Q_{IND\,(\text{esfera radio } b)}^{TOT} = 0$ porque se hacen nulos todos los coeficientes

(d) Aquí podemos usar superposición del siguiente modo:



① Ya se ha resuelto en el paso anterior por separación de variables

②

$$0<r<a \quad \phi = \frac{Q_1}{a} + \frac{Q_2}{b} + C_1 \rightarrow \phi = V_1$$

$$a<r<b \quad \phi = \frac{Q_1}{r} + \frac{Q_2}{b} + C_2 \rightarrow \phi = \frac{V_1 \cdot ab}{(b-a)}\left(\frac{1}{r} - \frac{1}{b}\right) = \frac{V_1 \cdot a}{(b-a)}\frac{(b-r)}{r}$$

$$b<r \quad \phi = \frac{Q_1}{r} + \frac{Q_2}{r} + C_3 \rightarrow \phi = 0$$

$$\phi(r\to\infty) = 0 \rightarrow C_3 = 0 \qquad \phi(b) = 0 = \frac{Q_1}{b} + \frac{Q_2}{b} \rightarrow Q_2 = -Q_1$$

$$\phi(r=b) = 0 = \frac{Q_1}{b} + \frac{Q_2}{b} + C_2 = \frac{Q_1}{b} - \frac{Q_1}{b} + C_2 \rightarrow C_2 = 0$$

$$(r=a) = V_1 = \frac{Q_1}{a} + \frac{Q_2}{b} + C_2 = \frac{Q_1}{a} - \frac{Q_1}{b} + C_2$$

$$V_1 = Q_1\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$Q_1 = \frac{V_1 \cdot a \cdot b}{(b-a)}$$

$$\phi(r=a) = V_1 = \frac{Q_1}{a} + \frac{Q_2}{b} + C_1 = \frac{Q_1}{a} - \frac{Q_1}{b} + C_1 = V_1$$

$$\underbrace{Q_1\left(\frac{1}{a} - \frac{1}{b}\right)}_{\text{ya era } V_1} + C_1 = V_1 \rightarrow C_1 = 0$$

③

$$0<r<a \quad \phi = \frac{Q_2}{b} + \frac{Q_1}{a} + C_1 \rightarrow \phi = 0$$

$$a<r<b \quad \phi = \frac{Q_2}{b} + \frac{Q_1}{r} + C_2 \rightarrow \phi = \frac{V_2\, b}{(b-a)} - \frac{V_2\, ab}{(b-a)r} = \frac{V_2 b[r-a]}{(b-a)r}$$

$$b<r \quad \phi = \frac{Q_2}{r} + \frac{Q_1}{r} + C_3 \rightarrow \phi = \frac{V_2 b^2}{(b-a)r} - \frac{V_2 ab}{(b-a)r} = \frac{V_2 b \cancel{(b-a)}}{\cancel{(b-a)}r}$$

$$\phi(r=b) = V_2 = \frac{Q_2}{b} + \frac{Q_1}{b} + C_3$$

$$\phi(r\to\infty) = 0 \rightarrow C_3 = 0 \rightarrow V_2 \cdot b - Q_2 = Q_1$$

$$\phi(r=b) = V_2 = \frac{Q_2}{b} + \frac{Q_1}{b} + C_2 \qquad V_2 = \cancel{Q_2/b} + Q_1/b - \cancel{Q_2/b} - Q_1/a$$

$$(r=a) = 0 = \frac{Q_2}{b} + \frac{Q_1}{a} + C_2 \rightarrow C_2 = -\frac{Q_2}{b} - \frac{Q_1}{a}$$

$$V_2 = Q_1\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\phi(r=a) = 0 = \frac{Q_2}{b} + \frac{Q_1}{a} + C_1$$

$$\frac{Q_2}{b} + \cancel{\frac{Q_1}{b}} = \cancel{\frac{Q_1}{b}} - \frac{Q_1}{a}$$

$$C_2 = -\frac{V_2 b}{b-a} + \frac{V_2 b}{b-a}$$

$$C_2 = 0$$

$$Q_1 = -\frac{a}{\cancel{b}}\frac{V_2 \cdot b^{\cancel{2}}}{(b-a)}$$

$$Q_2 = \frac{V_2 \cdot b^2}{(b-a)} \qquad V_2 = \frac{Q_2}{b}\left(1 - \frac{a}{b}\right) \qquad V_2 = \frac{Q_2}{b} - \frac{Q_2 a}{b^2} \leftarrow Q_1 = -\frac{Q_2}{b} \cdot a$$

Sumando las soluciones ② y ③ tenemos entonces

$$0<r<a \qquad \phi(r) = V_1$$

$$a<r<b \qquad \phi(r) = V_2 \cdot \frac{b}{r}\frac{(r-a)}{(b-a)} + V_1 \cdot \frac{a}{r} \cdot \frac{(b-r)}{(b-a)}$$

$$b<r \qquad \phi(r) = V_2 \cdot \frac{b}{r}$$

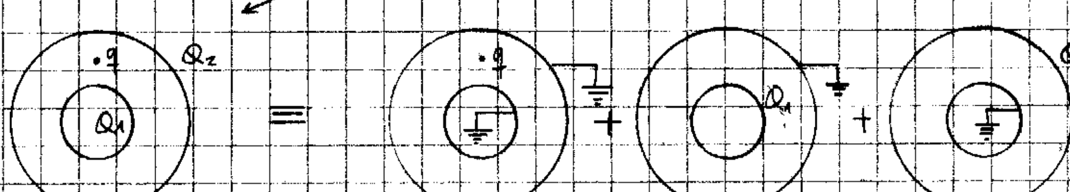
Sumando todos los casos ①+②+③ se tiene

$$\boxed{\begin{array}{lll} 0<r<a & \phi(r) = & V_1 \\ a<r<r_0 & \phi(r,\theta) = & \phi^{I}_{\ell_0}(r,\theta) + V_2 \frac{b}{r}\left(\frac{r-a}{b-a}\right) + V_1 \frac{a}{r}\left(\frac{b-r}{b-a}\right) \\ r_0<r<b & \phi(r,\theta) = & \phi^{II}_{\ell_0}(r,\theta) + V_2 \frac{b}{r}\left(\frac{r-a}{b-a}\right) + V_1 \frac{a}{r}\left(\frac{b-r}{b-a}\right) \\ r>b & \phi(r) = & V_2 \frac{b}{r} \end{array}}$$

(e)

En el caso anterior se puede construir el esquema final partiendo de ①; como q ya ha inducido σ en ambas esferas que compensan exactamente conectando a una batería V_1 y V_2 hace que se acumulen cargas Q_1 y Q_2 con σ uniformes dado que no se perturba a la σ que equilibró a q.

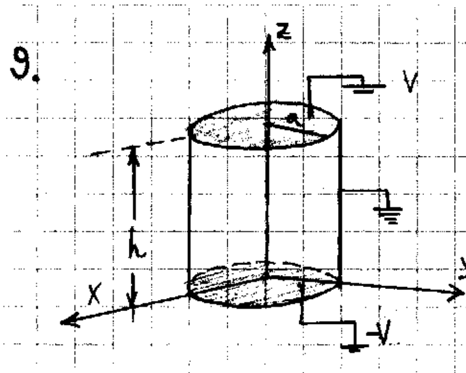
Para resolver ahora esta situación podemos usar la misma idea



①

Una vez que se equilibraron las cargas en ① tendremos una q'_a y q'_b inducidos sobre ambas esferas.
Luego le añadimos a la primera una carga $Q = Q_1 - q'_a$ que se distribuirá uniformemente sobre dicha esfera.
Después le añadimos $Q = Q_2 - q'_b$ a la segunda que se distribuirá también uniformemente, entonces hemos llegado a la situación que queríamos resolver.

9.

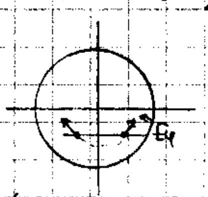


Simetría de rotación en φ $\Rightarrow$ simetría azimutal

$\vec{E} \neq \vec{E}(\varphi)$

reflexión en ZX

" " ZY



rotación y reflexión $\Rightarrow$ $E_\varphi = 0 \rightarrow \frac{\partial \phi}{\partial \varphi} = 0 \rightarrow \phi \neq \phi(\varphi)$

$$\phi(\rho, \varphi, z) = R(\rho) \cdot \underbrace{Q(\varphi)}_{cte.} \cdot Z(z)$$

Como no atravieso densidad de carga en $\hat{z}$ $\Rightarrow$ Z es exp. compleja

Como uso todo el rango de φ $(0 < \varphi < 2\pi)$ $\Rightarrow$ $\nu \in \mathbb{Z}$ pero $Q = (cte) \rightarrow \nu = 0$

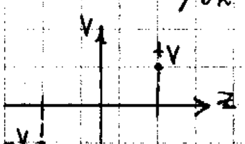
Uso Funciones de Bessel Modificadas

* Región I

utiliza eje $\hat{z}$ $\rightarrow$ $B_k^I = 0$ (pues Neumann revienta en $\rho = 0$) y $K_{\nu k}(k\rho)$ se comporta de Neumann

$$\phi_k^I(\rho, z) = \left[A_k^I \cdot I_0(k\rho) + B_k^I \cdot K_0(k\rho)\right]\left[C_k \cdot \cos(kz) + D_k \cdot \operatorname{sen}(kz)\right]$$

$$\phi_{0k}^I(\rho, z) = \sum_{k=1}^{\infty} A_k^I \cdot I_0(k\rho) \cdot \left[C_k \cdot \cos(kz) + D_k \operatorname{sen}(kz)\right]$$



Pero $\phi^I(a, z) = 0 \Rightarrow I_0(ka) = 0$

$-\frac{h}{2} < z < \frac{h}{2}$

$ka = x_{0n}$ ceros de Bessel

$k = \frac{x_{0n}}{a}$

$$\phi_k^I(\rho, z) = \sum_{n=1}^{\infty} I_0\left(\frac{x_n}{a}\rho\right) \cdot \left[C_n \cdot \cos\left(\frac{x_n z}{a}\right) + D_n \cdot \operatorname{sen}\left(\frac{x_n z}{a}\right)\right]$$

$$\phi_k^I(\rho, 0) = -V = \sum_{n=1}^{\infty} I_0\left(\frac{x_n \rho}{a}\right)\left[C_n\right]$$

$$\phi_k^I(\rho, h) = V = \sum_{n=1}^{\infty} I_0\left(\frac{x_n \rho}{a}\right)\left[C_n \cdot \cos\left(\frac{x_n h}{a}\right) + D_n \operatorname{sen}\left(\frac{x_n h}{a}\right)\right]$$

$$-C_n = C_n \cdot \cos\left(\frac{x_n h}{a}\right) + D_n \cdot \operatorname{sen}\left(\frac{x_n h}{a}\right)$$

$$\frac{-C_n\left(1 + \cos\left(\frac{x_n h}{a}\right)\right)}{\operatorname{sen}\left(\frac{x_n h}{a}\right)} = D_n$$

$$\phi_n^{I}(\rho,z) = \sum_{n=1}^{\infty} I_0\left(\frac{x_n\rho}{a}\right)\left[C_n \cdot \cos\left(\frac{x_n z}{a}\right) + D_n \cdot \mathrm{sen}\left(\frac{x_n z}{a}\right)\right]$$

$$-V = \sum_{n=1}^{\infty} I_0\left(\frac{x_n\rho}{a}\right) \cdot C_n$$

$$\int_0^a -V \cdot I_0\left(\frac{x_{n'}\rho}{a}\right)\rho \, d\rho = \int_0^a \sum_{n=1}^{\infty} I_0\left(\frac{x_n\rho}{a}\right) \cdot I_0\left(\frac{x_{n'}\rho}{a}\right)\rho \, C_n \, d\rho$$

$$-V \cdot \int_0^a I_0\left(\frac{x_{n'}\rho}{a}\right)\rho \, d\rho = \sum_{n=1}^{\infty} C_n \cdot \frac{a^2}{2}\left[I_1(x_n)\right]^2 \cdot \delta_{nn'}$$

$$-V \cdot \int_0^a I_0\left(\frac{x_{n'}\rho}{a}\right)\rho \, d\rho = C_{n'} \cdot \frac{a^2}{2}\left[I_1(x_{n'})\right]^2$$

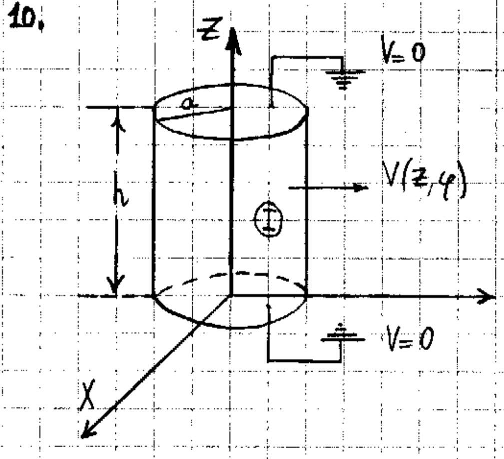
$$\boxed{C_n = -\frac{2V}{a^2} \frac{1}{\left[I_1(x_n)\right]^2} \int_0^a I_0\left(\frac{x_n\rho}{a}\right) \cdot \rho \, d\rho}$$

$$\boxed{D_n = \frac{2V}{a^2} \cdot \frac{1}{\left[I_1(x_n)\right]^2}\left(\int_0^a I_0\left(\frac{x_n\rho}{a}\right)\rho \, d\rho\right) \frac{\left(1 + \cos\left[x_n h/a\right]\right)}{\mathrm{sen}\left(x_n h/a\right)}}$$

$$\boxed{\phi_{inside}(\rho,z) = \sum_{n=1}^{\infty} I_0\left(\frac{x_n\rho}{a}\right) \cdot \left[C_n \cdot \cos\left(\frac{x_n z}{a}\right) + D_n \cdot \mathrm{sen}\left(\frac{x_n z}{a}\right)\right]}$$

Potencial en la zona interna del cilindro

10.



Tapas a tierra; lateral con $V(z,\varphi)$

No hay simetría de rotación en φ
y no hay simetría de reflexión en planos

Aún así tenemos condiciones periódicas en las tapas (en $\hat{z}$) $\Rightarrow$ $Z \to$ exp. complejas

$R \to I_\nu + K_\nu$

Como usamos todo el rango en $\varphi \to \nu \in \mathbb{Z}$

$Q \to$ exp. complejas

* Región (I) (zona interna al cilindro)

$$\phi^{I}(\rho,\varphi,z) = \left[A\, I_\nu(k\rho) + B\, K_\nu(k\rho)\right]\left[C\, \cos(kz) + D\, \mathrm{sen}(kz)\right].$$
$$\left[E\, \cos(\nu\varphi) + F\, \mathrm{sen}(\nu\varphi)\right]$$

Usamos el eje $z \to K_\nu$ no sirve $\to B=0$

$$\phi^{I}(a,\varphi,z) = V(\varphi,z)$$

$$\left.\begin{array}{l}\phi^{I}(\rho,\varphi,0) = 0\\ \phi^{I}(\rho,\varphi,h) = 0\end{array}\right\} \longrightarrow$$

Periodicidad en las tapas
$\mathrm{sen}(kh) = 0$ si $kh = n\pi \to k = \frac{n\pi}{h}$
$C=0$

el subíndice k se reemplaza por el n

$$\phi^{I}_{n\nu}(\rho,\varphi,z) = A^{I}_{n\nu}\, I_\nu\left(\frac{n\pi}{h}\rho\right) \mathrm{sen}\left(\frac{n\pi}{h}z\right)\left[E_{n\nu}\cos(\nu\varphi) + F_{n\nu}\,\mathrm{sen}(\nu\varphi)\right]$$

* Redefino constantes

$$\phi^{I}(\rho,\varphi,z) = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} I_\nu\left(\frac{n\pi}{h}\rho\right) \mathrm{sen}\left(\frac{n\pi z}{h}\right)\left[E^{I}_{n\nu}\cos(\nu\varphi) + F^{I}_{n\nu}\,\mathrm{sen}(\nu\varphi)\right]$$

hay metida una serie de Fourier de senos

$$V(\varphi,z) = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} I_\nu\left(\frac{n\pi a}{h}\right) \mathrm{sen}\left(\frac{n\pi z}{h}\right)\underbrace{\left[E^{I}_{n\nu}\cos(\nu\varphi) + F^{I}_{n\nu}\,\mathrm{sen}(\nu\varphi)\right]}_{f(\varphi)}$$

$$\int_0^h dz\, \mathrm{sen}\left(\frac{n'\pi z}{h}\right) \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} I_\nu\left(\frac{n\pi a}{h}\right) \mathrm{sen}\left(\frac{n\pi z}{h}\right)\left[f(\varphi)\right] = \int_0^h V(\varphi,z)\, dz\, \mathrm{sen}\left(\frac{n'\pi z}{h}\right)$$

$$\sum_{\nu=0}^{\infty}\sum_{n=1}^{\infty} I_\nu\left(\frac{n\pi a}{h}\right)\left[f(\varphi)\right]\underbrace{\int_0^h dz\, \mathrm{sen}\left(\frac{n'\pi z}{h}\right) \mathrm{sen}\left(\frac{n\pi z}{h}\right)}, = \int_0^h V(\varphi,z)\, \mathrm{sen}\left(\frac{n'\pi z}{h}\right) dz$$

$$\sum_{\nu=0}^{\infty}\sum_{n=1}^{\infty} I_\nu\left(\frac{n\pi a}{h}\right)\left[f(\varphi)\right] \frac{h}{2}\,\delta(n-n') = \int_0^h V(\varphi,z)\, \mathrm{sen}\left(\frac{n'\pi z}{h}\right) dz$$

$$\sum_{\nu=0}^{\infty} I_\nu\left(\frac{n'\pi a}{h}\right)\left[E^{I}_{n'\nu}\cos(\nu\varphi)+F^{I}_{n'\nu}\operatorname{sen}(\nu\varphi)\right]\frac{h}{2}=\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)dz$$

Ahora nos quedó una serie de Fourier en ν con período 2π $\rightarrow$ Ahora aplicamos ortogonalidad

$$\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)E^{I}_{n'\nu}\cos(\nu\varphi)+h\,I_\nu\left(\frac{n'\pi a}{h}\right)F^{I}_{n'\nu}\operatorname{sen}(\nu\varphi)=\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)dz$$

$$\underbrace{\int_0^{2\pi}\operatorname{sen}(\nu'\varphi)\,d\varphi\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)E^{I}_{n'\nu}\cos(\nu\varphi)}_{=0}+\int_0^{2\pi}\operatorname{sen}(\nu'\varphi)\,d\varphi\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)F^{I}_{n'\nu}\operatorname{sen}(\nu\varphi)$$

$$\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)\pi\,\delta(\nu-\nu')F^{I}_{n'\nu}=\int_0^{2\pi}\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)dz\,\operatorname{sen}(\nu'\varphi)\,d\varphi$$

$$\frac{h}{2}I_{\nu'}\left(\frac{n'\pi a}{h}\right)\pi\,F^{I}_{n'\nu'}=\int_0^{2\pi}d\varphi\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)\operatorname{sen}(\nu'\varphi)\,dz$$

sacamos las primas

$$F^{I}_{n\nu}=\frac{2}{\pi\,h\,I_\nu\left(\frac{n\pi a}{h}\right)}\int_0^{2\pi}d\varphi\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n\pi z}{h}\right)\operatorname{sen}(\nu\varphi)\,dz$$

Para el otro coeficiente hacemos

$$\int_0^{2\pi}\cos(\nu'\varphi)\,d\varphi\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)E^{I}_{n'\nu}\cos(\nu\varphi)+\underbrace{\int_0^{2\pi}\cos(\nu'\varphi)\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)F^{I}_{n'\nu}\operatorname{sen}(\nu\varphi)\,d\varphi}_{=0}$$

$$\sum_{\nu=0}^{\infty}\frac{h}{2}I_\nu\left(\frac{n'\pi a}{h}\right)E^{I}_{n'\nu}\pi\,\delta(\nu-\nu')=\int_0^{2\pi}\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)dz\,\cos(\nu'\varphi)\,d\varphi$$

$$\frac{h}{2}I_{\nu'}\left(\frac{n'\pi a}{h}\right)E^{I}_{n'\nu'}\pi=\int_0^{2\pi}d\varphi\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n'\pi z}{h}\right)\cos(\nu'\varphi)\,dz$$

sacamos las primas

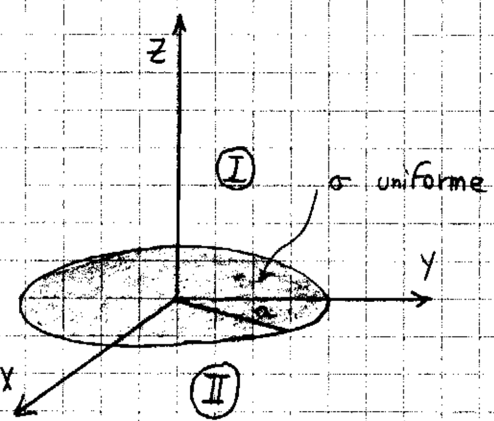
$$E^{I}_{n\nu}=\frac{2}{h\,\pi\,I_\nu\left(\frac{n\pi a}{h}\right)}\int_0^{2\pi}d\varphi\int_0^h V(\varphi,z)\operatorname{sen}\left(\frac{n\pi z}{h}\right)\cos(\nu\varphi)\,dz$$

$$\boxed{\phi^{I}(\rho,\varphi,z)=\sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}I_\nu\left(\frac{n\pi\rho}{h}\right)\operatorname{sen}\left(\frac{n\pi\rho}{h}\right)\left[E_{n\nu}\cos(\nu\varphi)+F_{n\nu}\operatorname{sen}(\nu\varphi)\right]}$$

▲ Potencial válido para el interior del cilindro.

11.

(a)



Simetría de rotación en φ
" " reflexión en XY
" " " " XZ
" " " " YZ

$E_\varphi = 0$ por rotación y reflexión $\Rightarrow \phi \neq \phi(\varphi)$

$$\phi(\rho, z) = R(\rho)\, Z(z)$$

hay un sigma en el plano XY $\quad \sigma(\rho) = \begin{cases} \sigma & \rho < a,\ z=0 \\ 0 & \rho > a,\ z=0 \end{cases}$

* En la dirección $\hat{z}$ atraviesa $\sigma \Rightarrow Z \propto$ exp. reales

* Uso $0 < \varphi < 2\pi \rightarrow \nu \in \mathbb{Z}$; simetría azimutal $\rightarrow \nu = 0 \rightarrow Q(\varphi) = B_{ok}$

* Separo en dos regiones donde vale $\nabla^2 \phi = 0$; ambas usan el eje $\hat{z}$ $\rightarrow$

$$R \propto J_\nu + N_\nu \quad \text{ésta la voy a anular}$$

* $\phi(z \to \infty) = 0 \rightarrow Z \to 0$ si $z \to \infty \rightarrow Z \propto e^{-kz}$

$$\phi^I_{\nu k} = A_{k\nu} J_\nu(k\rho) \left[C_{k\nu} \cdot e^{kz} + D_{k\nu} \cdot e^{-kz} \right] \cdot B_{ok}$$

$$\phi^I_{ko} = A_{ko} J_o(k\rho) \cdot E_{ko} e^{-kz} \underset{\substack{\text{redefiniendo} \\ \text{constantes}}}{=} A^I_{ko} \cdot J_o(k\rho)\, e^{-kz}$$

Como no hay condición de discretización en $k \rightarrow$

$$\phi^I(\rho, z) = \int_0^\infty A^I_{ko} \cdot J_o(k\rho)\, e^{-kz} \cdot dk$$

Análogamente para región II se tendrá:

$$\phi^{II}(\rho, z) = \int_0^\infty A^{II}_{ko} \cdot J_o(k\rho)\, e^{kz} \cdot dk$$

* Continuidad de ϕ en $z=0$

$$\phi_I(\rho, 0) = \phi_{II}(\rho, 0) \rightarrow$$

$$\int_0^\infty A^I_{ko} \cdot J_o(k\rho)\, dk = \int_0^\infty A^{II}_{ko}\, J_o(k\rho) \cdot dk \quad \Rightarrow A^I_{ko} = A^{II}_{ko} \equiv A_k$$

* Salto del campo por la σ

$$\frac{\partial \phi_I}{\partial z} - \frac{\partial \phi_{II}}{\partial z}\Big|_{z=0} = \begin{cases} 4\pi\sigma & \text{si} \quad \rho<a \\ 0 & \text{si} \quad \rho>a \end{cases}$$

$$\int_0^\infty A_{k0}\, J_0(k\rho)\cdot k\; dk + \int_0^\infty A_{k0}\cdot J_0(k\rho)\cdot k\; dk = \begin{cases} 4\pi\sigma \\ 0 \end{cases}$$

$$2\int_0^\infty A_{k0}\cdot J_0(k\rho)\cdot k\, dk = \begin{cases} 4\pi\sigma \\ 0 \end{cases}$$

* Ortogonalidad para los coeficientes (con J de Bessel)

$$\int_0^\infty 2\int_0^\infty A_{k0}\, J_0(k\rho)\, k\, dk\, J_0(k'\rho)\cdot\rho\, d\rho = \int_0^a 4\pi\sigma\cdot J_0(k'\rho)\,\rho\, d\rho$$

$$\int_0^\infty A_{k0}\cdot 2\cdot k\cdot \int_0^\infty \rho\, J_0(k\rho)\, J_0(k'\rho)\, d\rho\; dk$$

$$\int_0^\infty A_{k0}\cdot 2\cdot \cancel{k}\,\frac{1}{\cancel{k}}\,\delta(k'-k)\; dk = 4\pi\sigma\int_0^a J_0(k'\rho)\,\rho\, d\rho$$

$$A_{k0}\cdot 2 = 4\pi\sigma\int_0^a J_0(k\rho)\,\rho\, d\rho$$

$$A_{k0} = 2\pi\frac{\sigma}{k^2}\int_0^{ka} J_0(x)\, x\, dx$$

$k\rho = x$

$d\rho = \frac{dx}{k}$

$$A_{k0} = \frac{2\pi\sigma}{k^2}\; x\cdot J_1(x)\Big|_0^{ka} = \frac{2\pi\sigma}{k^2}\, ka\, J_1(ka)$$

$$\boxed{\phi_I(\rho,z) = \int_0^\infty \frac{2\pi\sigma a}{k}\, J_1(ka)\cdot e^{-kz}\, J_0(k\rho)\; dk}$$

$$\boxed{\phi_{II}(\rho,z) = \int_0^\infty \frac{2\pi\sigma\cdot a}{k}\, J_1(ka)\; e^{kz}\, J_0(k\rho)\; dk}$$

Potencial para todo el espacio

* Cálculo del Campo

$$-\vec{\nabla}\phi = \vec{E} \Rightarrow$$

$$E_\rho^I = 2\pi\sigma a\int_0^\infty \frac{1}{k}\, J_1(ka)\cdot e^{kz}\,\frac{d}{d\rho}\left[J_0(k\rho)\right] dk$$

$$E_\varphi^I = 0$$

$$E_z^I = -2\pi\sigma a\int_0^\infty \frac{1}{\cancel{k}}\, J_1(ka)\cdot J_0(k\rho)\; e^{kz}\cdot\cancel{k}\; dk$$

$$E_\rho^{II} = 2\pi\sigma a \int_0^\infty \frac{1}{k} J_1(ka)\, e^{kz} \frac{d}{d\rho}\left[J_0(k\rho)\right] dk$$

$$E_\varphi^{II} = 0$$

$$E_z^{II} = 2\pi\sigma a \int_0^\infty J_1(ka)\, J_0(k\rho)\, e^{-kz}\, dk$$

(b)

Si $z \ll a \rightarrow \frac{z}{a} \ll 1$ $\qquad J_0(k\rho) = \sum_{j=0} \frac{(-1)^j}{j!\,\Gamma(j+1)} \left(\frac{k\rho}{2}\right)^{2j}$

$\Rightarrow$ Me puedo parar en $\rho = 0$ [origen] y tendré una representación de un plano ∞

$$E_z^{II}(\rho=0, z \ll a) \cong 2\pi\sigma a \int_0^\infty J_1(ka) \overbrace{J_0(0)}^{=1} e^{-kz}\, dk$$

$$\int_0^\infty J_1(ka)\, e^{-kz}\, dk$$

mediante un handbook $\quad a = z \quad b = a$

$$E_z^{II} \cong 2\pi\sigma a \left[\frac{\sqrt{z^2+a^2} - z}{a^2\sqrt{z^2+a^2}}\right]^1$$

con $n > -1$ (z que es uno)

$$E_z^{II} \cong 2\pi\sigma a \left(a \underbrace{\sqrt{\frac{z^2}{a^2}+1}}_{\to 0} - z\right) \frac{1}{a^2 \underbrace{\sqrt{\frac{z^2}{a^2}+1}}_{\to 0}}$$

$$E_z^{II} \cong 2\pi\sigma a \frac{(a - z)}{a^2} = 2\pi\sigma - 2\pi\sigma\left(\frac{z}{a}\right)$$

Para un plano ∞ el valor es $\vec{E} = 2\pi\sigma\,\hat{n}$, con lo cual es una aproximación decente

$$\boxed{E_z^{I} \cong 2\pi\sigma \left(1 + \frac{z}{a} + \underset{\text{muy chicos}}{\overset{\dots}{\text{términos}}}\right)}$$

Si $z \gg a \rightarrow \frac{a}{z} \ll 1$ $\rightarrow$ Me paro otra vez en $\rho = 0$ [origen] y me voy en z bien lejos

$$E_z^{I} \cong 2\pi\sigma a \left[\frac{z\sqrt{1+(a/z)^2} - z}{a^2\, z\sqrt{1+\frac{a^2}{z^2}}}\right]$$

$\sigma = \frac{Q}{\pi a^2}$

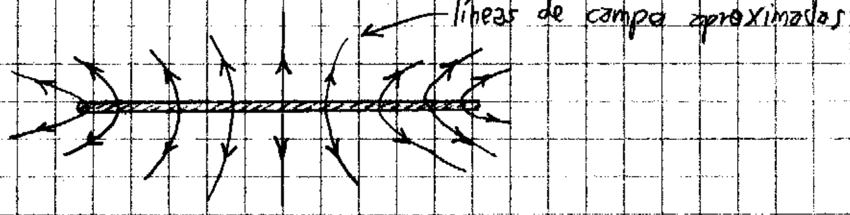
$$\cong \frac{2Q}{a} \cdot \frac{\cancel{z}\,(\cancel{1} + 1/2\, a^2/z^2 - \cancel{1})}{a\,\cancel{z}} \left[1 - \frac{1}{2}\frac{a^2}{z^2}\right]$$

$$\cong \frac{Q\cancel{2}}{\cancel{a}} \frac{1}{\cancel{a}} \left(\frac{1}{\cancel{2}}\frac{a^{\cancel{2}}}{z^2}\right)\left(1 - \frac{1}{2}\frac{a^2}{z^2}\right)$$

$$\boxed{E_z^{I} \cong \frac{Q}{z^2} - \frac{Q a^2}{2 z^4}}$$

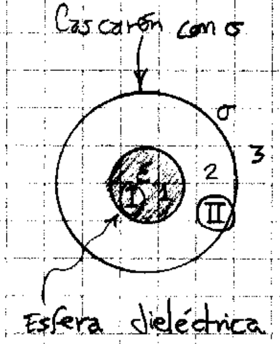
Aproximadamente presenta campo de forma carga puntual pues el término $\left(\frac{a}{z^2}\right)^2$ es muy pequeño y puede desecharse

(c) El disco no es conductor; si lo fuese la σ no podría ser uniforme sino que se acomodaría para dar V constante en la superficie.
Sería una σ que tendría un valor muy alto en los bordes



IV. Preguntas Conceptuales

4.



Se puede dividir en tres regiones en c/u de las cuales vale Laplace:

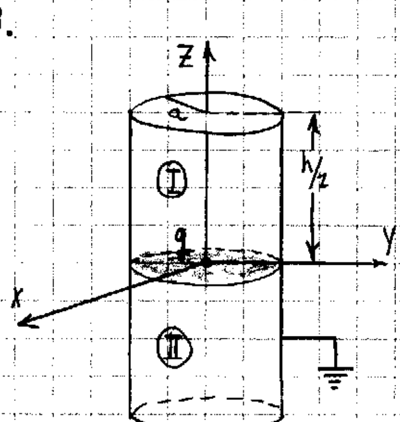
1. $\varepsilon \nabla^2 \phi = 0$
2. $\nabla^2 \phi = 0$
3. $\nabla^2 \phi = 0$

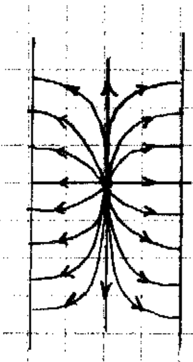
O si no puede pensarse en dos regiones donde vale Laplace y Poisson

$$\text{I} \quad \varepsilon \nabla^2 \phi = 0$$
$$\text{II} \quad \nabla^2 \phi = -4\pi \rho = -4\pi \cdot \sigma \cdot \delta(r - r_0)$$

De la segunda forma permite ser resuelto por separación de variables sumando para la zona II una solución de Poisson a la solución de Laplace

3.





1. La carga q induce carga en las paredes de la caja cilíndrica a tierra con lo cual debería tenerse en cuenta para la resolución por Poisson. Ello implicaría conocer la distribución de carga $\sigma(\vec{r})$ lo cual es muy poco práctico.

   Lo mas recomendable es dividir en dos zonas I, II en c/u de las cuales vale Laplace y poner como contornos la tapa central con $\sigma = q\,\delta(\rho)$ y el lateral y la otra tapa con $V = 0$, procediendo en forma análoga en la otra región