②. $\displaystyle\int_0^{\infty} \frac{\cos x \, . \, dx}{(x^2+1)(x^2+9)}$

$\left|\dfrac{\cos x}{(x^2+1).(x^2+9)}\right| \leqslant \left|\dfrac{1}{x^4}\right| = \dfrac{1}{x^4} \quad \Rightarrow \quad \int | \; | dx \leqslant \int \frac{1}{x^4} . dx$

$$\left.\begin{array}{r} |x^4+x^2+9x^2+9| = |x^4+10x^2+9| \leqslant |x^4|+10|x^2|+9 \\ |x^4| \leqslant |x|^4+10|x|^2+9 \\ \dfrac{1}{|x^4|} \geqslant \dfrac{1}{|x^4|+10|x|^2+9} \end{array}\right]$$

pero como $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converge

la integral $\displaystyle\int \frac{1}{x^4} . dx$ debe converger

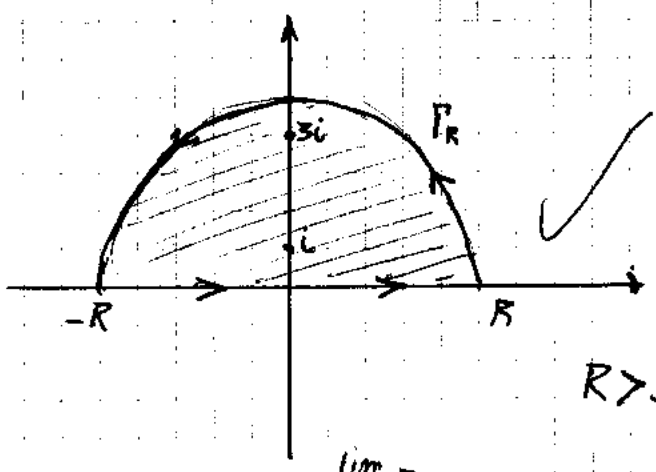
Considero $f(z) = \dfrac{e^{iz}}{(z^2+1)(z^2+9)}$

$\displaystyle\lim_{|z|\to\infty} z . f(z) = \lim_{|z|\to\infty} \frac{z \, e^{iz}}{(z^2+1)(z^2+9)} = 0 \rightarrow$ puedo usar $f(z)$ e integrar en algún recinto Γ cuya parte que camina sobre $\mathbb{R}$ tomo todo x real.

$(z^2+1)(z^2+9) = (z-i)(z+i)(z+3i)(z-3i) \longrightarrow$

$(z^2-zi+zi-i^2)(z^2+3iz-3iz-9i^2)$

$f(z) = \dfrac{e^{iz}}{(z-i)(z+i)(z+3i)(z-3i)}$ tiene polos simples en $i, -i, 3i, -3i$



$\Gamma = \overline{-RR} + \Gamma_R$

curva sobre la cual f es holomorfa y lo mismo en su interior salvo $z=i$, $z=3i$

$\Rightarrow$

$R > 3$

$\displaystyle\int_\Gamma f(z).dz \overset{\lim_{R\to\infty}}{=} \int_{-R}^{R} f(x).dx + \int_{\Gamma_R} f(z).dz = 2\pi i . \left[\mathrm{Res}(f,i) + \mathrm{Res}(f,3i) \right]$

/ $\displaystyle\mathrm{Res}(f,i) = \lim_{z\to i} \frac{\cancel{(z-i)} \, e^{iz}}{\cancel{(z-i)}(z+i)(z+3i)(z-3i)} = \frac{e^{i^2}}{2i(4i)(-2i)} = \frac{e^{i^2}}{.16i} = \frac{e}{16i}$

$-8(-2i)$

$16i$ $=$

$\displaystyle\mathrm{Res}(f,3i) = \lim_{z\to 3i} \frac{\cancel{(z-3i)} . e^{iz}}{(z-i)(z+i)(z+3i)\cancel{(z-3i)}} = \frac{e^{3i^2}}{2i(4i)(6i)} = \frac{e^{-3}}{(-48i)}$

$-8 . (6i)$ $=$

$$\lim_{R\to\infty}\left(\int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz\right) = 2\pi i \left(\frac{e^{-1}}{16i} - \frac{e^{-3}}{48i}\right)$$

$$= \frac{2\pi i}{16 i}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

$$= \frac{\pi}{8}\left[\frac{1}{e} - \frac{1}{3e^3}\right] \checkmark$$

* integral en Γ_R

$$\int_{\Gamma_R} f(z)\,dz = \int_{\Gamma_R} \frac{e^{iz}\,dz}{(z^2+1)(z^2+9)} = \int_0^{\pi} \frac{e^{iRe^{i\theta}} \cdot R e^{i\theta} i}{(R^2 e^{i2\theta}+1)(R^2 e^{i2\theta}+9)}\,d\theta \checkmark$$

Γ_R

$z(\theta) = R\,e^{i\theta} \qquad 0 < \theta < \pi$

$z'(\theta) = R\,e^{i\theta}\,i$

$$= \underbrace{\int_0^{\pi} \frac{\left(e^{iRe^{i\theta}}\right) R\,e^{i\theta}\,i\,d\theta}{2\,(R^2 e^{i2\theta}+1)(R^2 e^{i2\theta}+9)}}_{=\,\boxminus}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$|\boxminus| = \left|\frac{\left[e^{iR(\cos\theta + i\,\mathrm{sen}\theta)}\right] R}{2\,(R^4 e^{i4\theta} + R^2 e^{i2\theta} + 9R^2 e^{i2\theta} + 9)}\right| \checkmark$$

$$\left|\frac{\left(e^{iR\cos\theta}\, e^{-R\,\mathrm{sen}\theta}\right) R}{(R^4 e^{i4\theta} + 10R^2 e^{i2\theta} + 9)\,2}\right| \leq \frac{R\left(\left|e^{-R\,\mathrm{sen}\theta}\right|\right)}{2\,|R^4 - 10R^2 - 9|}$$

$\geq |R^4| - |10R^2 e^{i2\theta} + 9| \geq |R^4 - (|10R^2| - |9|)| \geq R^4 - 10R^2 + 9$

$$|R^4 - 10R^2 + 9| \leq |R^4 e^{i4\theta} + 10R^2 e^{i2\theta} + 9| \leq |R^4| + 10|R^2| + 9 \checkmark$$

$10 = 10 \cdot 4 \cdot 9 \cdot 1$ pero a partir de cierto R_0 vale $|R^4 - 10R^2 + 9| > |R^3| \rightarrow$

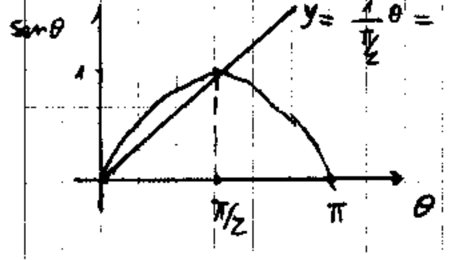
$$\frac{1}{|R^4 - 10R^2 + 9|} < \frac{1}{R^3} \checkmark$$

$$\leq \frac{\cancel{R}\,e^{-R\,\mathrm{sen}\theta}}{2 \cdot R^{\cancel{3}\,2}} = \frac{|e^{-R\,\mathrm{sen}\theta}|}{2R^2} \leq \frac{e^{-\frac{2\theta}{\pi}R}}{2R^2} \quad \text{con } 0 < \theta < \pi/2$$

$\mathrm{sen}\theta \geq \frac{2\theta}{\pi}$ si $\theta \in (0, \pi/2) \rightarrow$

$$-\mathrm{sen}\theta \leq -\frac{2\theta}{\pi}$$

$$e^{-\mathrm{sen}\theta} \leq e^{-\frac{2\theta}{\pi}}$$



Luego: $\checkmark$

$$\int_0^{\pi} |\sqcup| \, d\theta \leq \int_0^{\pi} \frac{e^{-R \cdot \operatorname{sen}\theta}}{2R^2} \, d\theta = \frac{1}{2R^2} \int_0^{\pi} e^{-R \cdot \operatorname{sen}\theta} \, d\theta = \frac{1 \cdot \cancel{2}}{\cancel{2}R^2} \int_0^{\pi/2} e^{-R \cdot \operatorname{sen}(2\varphi)} \, d\varphi$$

Se puede resolver mucho más fácil !

Sea $\varphi = \theta/2$, $d\varphi = \frac{1}{2} \cdot d\theta$ $\rightarrow$ $0 \leq \theta \leq \pi$, $0 \leq 2\varphi \leq \pi \rightarrow 0 \leq \varphi \leq \pi/2$

$$\int_0^{\pi} |\sqcup| \, d\theta \leq \frac{1}{R^2} \int_0^{\pi/2} e^{-R \cdot \operatorname{sen}(2\varphi)} \, d\varphi \leq \frac{1}{R^2} \int_0^{\pi/2} e^{-\frac{2}{\pi} \cdot R \cdot 2\varphi} \, d\varphi$$

$-\frac{4R}{\pi}\varphi = u$

$-\frac{4R}{\pi} d\varphi = du$

$d\varphi = \frac{-\pi}{4R} du$

$$\leq \frac{1}{R^2} \cdot \frac{-\pi}{4R} \int_0^{-R2} e^u \, du =$$

$$= \frac{-\pi}{4R^3} \cdot e^u \Big]_0^{-2R} = \frac{-\pi}{4R^3}\left(e^{-2R} - e^0\right)$$

Luego:

$$\lim_{R\to\infty} \int_{-R}^{R} f(x) \, dx + \lim_{R\to\infty} \frac{-\pi}{4R^3}\left(e^{-2R} - e^0\right) = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+1)(x^2+9)} \, dx = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

$$\int_{-\infty}^{+\infty} \frac{\cos x \, dx}{(x^2+1)(x^2+9)} + i \cdot \int_{-\infty}^{+\infty} \frac{\operatorname{sen} x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

igualando partes $\Re e$ e $\Im m$ se tiene

$$\int_{-\infty}^{+\infty} \frac{\cos x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

pero esta integral tiene integrando par $\Rightarrow$

$$2\int_0^{+\infty} \frac{\cos x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right)$$

$$\boxed{\int_0^{+\infty} \frac{\cos x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16}\left(\frac{1}{e} - \frac{1}{3e^3}\right)}$$