5.

$$f(z) = \frac{1}{z(e^z-1)}$$

$z = 0$

$e^z = 1 \iff$

$e^{x+iy} = e^x[\cos y + i\,\mathrm{sen}\, y] = 1$

$e^x \cos y + i e^x \mathrm{sen}\, y = 1$

$e^x\, \mathrm{sen}\, y = 0 \rightarrow \mathrm{sen}\, y = 0 \quad y = k\pi$

$e^x \cos y = 1$

$e^x \cos(k\pi) = 1$

$\downarrow \quad x = 0$

1 (k par) $\quad -1$ (k impar) $\qquad \boxed{z = 2\underbrace{k\pi i}_{y}}$

singularidades en $\mathbb{C}$

$z = 0, \quad z = 2k\pi i \quad (k \in \mathbb{Z}_{\neq 0})$

$k \neq 0$

$z = 2k\pi i$

$$f(z) = \frac{1/z}{(e^z-1)} \xrightarrow[z \to 2k\pi]{} \infty \quad \Rightarrow \quad z = 2k\pi \ (k \neq 0) \text{ son polos}$$

$$\frac{1}{f(z)} = z\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) = z\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n!} = z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \frac{z^5}{4!} + \dots$$

$$= z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)$$

$$f(z) = \frac{1}{\sum_{n=1}^{\infty}\frac{z^{n+1}}{n!}} = \frac{1}{z^2(\ g[z]\ )} = \frac{1/g(z)}{z^2} \longrightarrow \boxed{z = 0 \text{ es polo de orden } 2}$$

$g(z=0) = 1$, analítica en torno a $z = 0$

$$f(z) = \frac{1}{z\left(e^{z+2\pi k i - 2\pi k i} - 1\right)} = \frac{1/z}{e^{2k\pi i}\cdot e^{z - 2k\pi i} - 1} = \frac{1/z}{\left(\sum_{n=0}^{\infty}\frac{(z-2k\pi)^n}{n!} - 1\right)} = \frac{1/z}{\sum_{n=1}^{\infty}\frac{(z-2k\pi)^n}{n!}}$$

$$= \frac{(1/z)\left(1/\sum_{n=1}^{\infty}(z-2k\pi)^{n-1}/n!\right)}{(z-2k\pi i)} \longrightarrow \equiv h(z)\ ;\ h(2k\pi) \neq 0 ; \text{ analítica en torno a } z = 2k\pi$$

$$\boxed{z = 2k\pi i \quad \text{es polo simple}}$$

$$\mathrm{Res}\left(\frac{1}{z(e^z-1)}, 2k\pi i\right) = \lim_{z \to 2k\pi} \frac{(z-2k\pi i)}{z(e^z-1)} \underset{\text{l'hopital}}{=} \lim_{z\to 2k\pi i} \frac{1}{(e^z-1) + z(e^z)} = \frac{1}{2k\pi i\, e^{2k\pi i}} = \frac{1}{2k\pi i}$$

$$f(z) = \boxed{\frac{1}{2k\pi i(z-2k\pi i)}} + \sum_{n=0}^{\infty} a_n \cdot (z-2k\pi)^n$$

parte singular de $f(z)$ en $z = 2k\pi$ $(k \in \mathbb{Z} \neq 0)$

$$\operatorname{Res}\left(\frac{1}{z(e^z-1)}, 0\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^{\cancel{2}}}{\cancel{z}(e^z-1)}\right) = \lim_{z\to 0} \frac{1(e^z-1) - z\cdot e^z}{[e^z-1]^2} \overset{L'H}{=} \lim_{z\to 0} \frac{e^z - ze^z - e^z}{2(e^z-1)} \overset{L'H}{=} \lim_{z\to 0} \frac{-ze^z - e^z}{2e^z} = \boxed{\frac{-1}{2}}$$

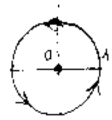
de la serie de Laurent en $0<|z|<2\pi$

$$f(z) = \frac{a_{(-2)}}{z^2} - \frac{1/2}{z} + \sum_{n=0}^{\infty} a_n (z)^n$$

$$a_{-2} = \frac{1}{2\pi i}\int_{C^+} \frac{\frac{1}{z(e^z-1)}}{(z-z_0)^{-2+1}}\cdot dz = \frac{1}{2\pi i}\int_{C^+} \frac{1}{e^z-1}\cdot dz$$

$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{e^z-1}\cdot dz = a_{-2}$$

$$\int_{|z|=1} \frac{1}{e^z-1}\cdot dz = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{e^z-1}, 0\right)$$



$$\int_{|z|=1} \frac{1}{e^z-1}\cdot dz = 2\pi i$$

$$\frac{1}{\sum_{n=0}^{\infty}\frac{z^n}{n!} - 1} = \frac{1}{\sum_{n=1}^{\infty}\frac{z^n}{n!}} = \frac{1}{z \cdot \sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}}$$ $\neq 0$ analítica en $z=0$

$z=0 \to$ polo simple

$$\operatorname{Res}\left(\frac{1}{e^z-1}, 0\right) = \lim_{z\to 0} \frac{z}{e^z-1}$$

$$\frac{1}{e^z} = \boxed{1}$$

Parte principal de f en $z=0$ ↘ $= \boxed{\frac{1}{z^2} - \frac{1/2}{z}}$