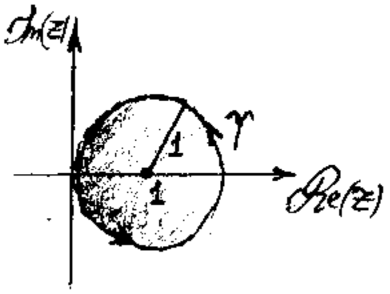
Ejercicio 3

$\gamma = \{z \in \mathbb{C} : |z-1| = 1\}$ $\qquad \displaystyle\int_\gamma \frac{e^{2z}}{(z^2-1)^2}\,dz$



Vamos a descomponerlo mediante fracciones simples:

$$(z^2-1)^2 = (z+1)^2(z-1)^2$$

$$\frac{1}{(z^2-1)^2} = \frac{1}{(z+1)^2\cdot(z-1)^2}$$

$$\frac{1}{(z^2-1)^2} = \frac{1}{(z+1)^2(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-1)} + \frac{D}{(z-1)^2}$$

$$1 = A(z+1)(z-1)^2 + B(z-1)^2 + C(z+1)^2(z-1) + D(z+1)^2$$

$$1 = A(z^2-1)(z-1) + B(z^2-2z+1) + C(z^2-1)(z+1) + D(z^2+2z+1)$$

$\cancel{Az}$

$$1 = A(z^3-z-z^2+1) + B(z^2-2z+1) + C(z^3-z+z^2-1) + D(z^2+2z+1)$$

$0 = z^3(A+C) \quad \rightarrow \quad A = -C$

$0 = z^2(-A+C+B+D) \qquad A = -A+B+D \rightarrow A = \frac{B+D}{2}$

$0 = z(-A-2B-C+2D) \qquad \frac{B}{2}+\frac{D}{2} = -2B + \frac{B}{2} + \frac{D}{2} + 2D \rightarrow \quad B = D$

$1 = 1(A+B-C+D) \rightarrow$

$1 = 2A + B + D$
$1 = B+D+B+D$
$1 = 2(B+D)$
$1 = 4A$
$\boxed{\frac{1}{4} = A}$ $\qquad \frac{1}{2} = B+D \rightarrow 2B = \frac{1}{2}$ $\qquad \boxed{C = -\frac{1}{4}}$

$\boxed{B = \frac{1}{4} = D}$

$$\int_\gamma \frac{e^{2z}}{(z^2-1)^2}dz = \frac{1}{4}\underbrace{\int_\gamma \frac{e^{2z}}{z+1}dz}_{1} + \frac{1}{4}\underbrace{\int_\gamma \frac{e^{2z}}{(z+1)^2}dz}_{2} - \frac{1}{4}\underbrace{\int_\gamma \frac{e^{2z}}{z-1}dz}_{3} + \frac{1}{4}\underbrace{\int_\gamma \frac{e^{2z}}{(z-1)^2}dz}_{4}$$

1. $\dfrac{e^{2z}}{z+1}$ es holomorfa en γ y en su interior $\Rightarrow \displaystyle\int_\gamma \frac{e^{2z}}{z+1}dz = 0$

2. $\dfrac{e^{2z}}{(z+1)^2}$ es holomorfa en γ y en su interior $\Rightarrow \displaystyle\int_\gamma \frac{e^{2z}}{(z+1)^2}dz = 0$

③ $-\frac{1}{4}\int_\gamma \frac{e^{zz}}{z-1}\,dz = -\frac{1}{2}\pi . e^{2} i$

e^{zz} es holomorfa en γ y en su interior, γ está recorrida en sentido positivo $z=1$ es interior a γ $\Rightarrow$ se puede aplicar fórmula de Cauchy

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

$$e^{zz}(z_0=1) = \frac{1}{2\pi i}\int_\gamma \frac{e^{zz}}{z-1}\,dz \rightarrow -\frac{1}{4}\int \frac{e^{zz}}{z-1}dz = -\frac{e^{2}.2\pi i}{4}$$

④ $\frac{1}{4}\int \frac{e^{zz}}{(z-1)^2}\,dz = e^{2}\pi i$

Con las mismas condiciones que en la integral anterior puede emplearse la fórmula de Cauchy derivada

$$f'(z) = \frac{1!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}\,dw$$

$$e^{zz}.z(z=1) = \frac{1}{2\pi i}\int_\gamma \frac{e^{zz}}{(z-1)^2}\,dz \rightarrow \frac{1}{4}\int_\gamma \frac{e^{zz}}{(z-1)^2}dz = \frac{2e^{2}.2\pi i}{4}$$

$$\therefore \quad \int_\gamma \frac{e^{zz}}{(z^2-1)^2}\,dz = -\frac{e^{2}\pi i}{2} + e^{2}\pi i = \boxed{\frac{1}{2}e^{2}\pi i}$$