PRACTICA 9

1. i) $\phi(x,y) = \langle x,y\rangle = 2x_1y_1 + 3x_2y_1 - x_2y_2 + 3x_1y_2$

$\mathbb{V} = \mathbb{R}^2 \Rightarrow \quad A = \begin{pmatrix} \langle e_1,e_1\rangle & \langle e_1,e_2\rangle \\ \langle e_2,e_1\rangle & \langle e_2,e_2\rangle \end{pmatrix}$

$e_1 = (1,0) \quad e_2 = (0,1)$

$\langle x,x\rangle = 2x_1^2 + 3x_2x_1 - x_2^2 + 3x_1x_2$
$2x_1^2 + 6x_2x_1 - x_2^2$
$x_1^2 + x_1^2 + 2x_2x_1 + x_2^2 - 2x_2^2$
$(x_1+x_2)^2 + x_1^2 - 2x_2^2$
$(x_1^2 + 2x_1x_2 + x_2^2)$ $-3x_2^2$
$2(x_1+x_2)^2 + 2x_1x_2 - 3x_2^2$

$A = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}$ A no es definida positiva

$\boxed{\phi(x,y) \text{ no es producto interno}}$

ii) $\phi(x,y) = \langle x,y\rangle = x_1\cdot y_1 + x_2\cdot y_1 + 2x_2y_2 - 3x_1y_2$

$\mathbb{V} = \mathbb{R}^2 \qquad e_1 = (1,0) \qquad e_2 = (0,1)$

$A = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$ A no es definida positiva

$\boxed{\phi(x,y) \text{ no es producto interno}}$

iii) $\phi(x,y) = \langle x,y\rangle = 2x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1$

$\mathbb{V} = \mathbb{R}^2$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad 2>0 \quad 1>0 \qquad |A| = 3 > 0$ A def. positiva

$\boxed{\phi(x,y) \text{ es producto interno sobre } \mathbb{R}^2}$

$\mathbb{V} = \mathbb{C}^2 \qquad \underset{\text{(escalares)}}{K = \mathbb{C}} \qquad B = \{(1,i),(1,-i)\} \to$ base de $\mathbb{C}^2$ $(K = \mathbb{C})$

$A = \begin{pmatrix} \overbrace{2+i^2-i-i}^{1-2i} & 2-i^2+i-i \\ 2-i^2-i+i & \underbrace{2+i^2+i+i}_{1+2i} \end{pmatrix}$

$|A| = (1-2i).(1+2i) - 3.3$
$= 1-2i+2i-4i^2 - 9$
$= 5-9 = -4$

no está definida $\oplus$

$\boxed{\phi(x,y) \text{ no es producto interno sobre } \mathbb{C}^2}$

iv) $\phi(x,y) = \langle x,y\rangle = 2x_1\bar{y}_1 + x_2\bar{y}_2 - x_1\bar{y}_2 - x_2\bar{y}_1$

$\mathbb{V} = \mathbb{C}^2 \qquad K = \mathbb{C} \qquad B = \{(1,i),(1,-i)\}$

Probamos con la propiedad de simetría

$\langle y,x\rangle = 2y_1\bar{x}_1 + y_2.\bar{x}_2 - y_1.\bar{x}_2 - y_2.\bar{x}_1$

$\overline{\langle y,x\rangle} = 2.\bar{y}_1x_1 + \bar{y}_2x_2 - \bar{y}_1.x_2 - \bar{y}_2.x_1 = \langle x,y\rangle$

$\langle x,x\rangle = 2x_1\bar{x}_1 + x_2\bar{x}_2 - x_1\bar{x}_2 - x_2\bar{x}_1$

2. i) $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

a) $\phi(x,y) = \langle x,y\rangle = x_1 \cdot y_1 - 2x_1y_2 - 2x_2y_1 + 6x_2y_2$

$\langle \lambda_1 x + \lambda_2 z, y\rangle$

$= (\lambda_1 x_1 + \lambda_2 z_1) \cdot y_1 - 2(\lambda_1 x_1 + \lambda_2 z_1) y_2 - 2(\lambda_1 x_2 + \lambda_2 z_2) y_1 + 6(\lambda_1 x_2 + \lambda_2 z_2) y_2$

$= \lambda_1 (x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 6x_2 y_2) + \lambda_2 (z_1 y_1 - 2z_1 y_2 - 2z_2 y_1 + 6z_2 y_2)$

$= \lambda_1 \langle x,y\rangle + \lambda_2 \langle z, y\rangle$

propiedades I-III

$\boxed{\phi \text{ Es un producto interno}}$

$\phi(x,x) = x_1^2 - 2x_1x_2 - 2x_2x_1 + 6x_2^2 \geq 0$

$x_1^2 - 4x_2x_1 + 6x_2^2 \geq x_1^2 - 4x_2x_1 + 4x_2^2 + 2x_2^2$

propiedad IV

$(x_1 - x_2 \cdot 2)^2 + 2x_2^2 \geq 0$

$(x_1 - x_2 2)^2 + 2x_2^2 = 0$

$\Leftrightarrow \quad x_1 = x_2 = 0$

b) $B_{\mathbb{R}^2} = \{(1,0),(0,1)\}$

ortogonal para Φ

$V_1' = (1,0)$ $\quad \|V_1\| = \sqrt{\langle V_1, V_1\rangle} = 1$ con Φ

$V_2' = (0,1) - \left(\frac{(0,1).(1,0)}{\|V_1'\|}\right) \cdot (1,0)$

$(0,1) - \left(\frac{0 - 2.0 - 2}{1}\right) \cdot (1,0) = (0,1) - (-2)(1,0) = (+2,1)$

$\boxed{B'_{\mathbb{R}^2} = \{(1,0),(+2,1)\}}$ ortogonal

$\boxed{B'_{\mathbb{R}^2} = \{(1,0),(+2/\sqrt{2}, 1/\sqrt{2})\}}$ normalizado

$\|(2,1)\| = 4 - 2.2.1 - 2.1.2 + 6.1.1$

$\|(1,0)\| = 1 \quad 2 - 4 \quad -4 \quad +6$

ii) $B_{\mathbb{C}^2} = \{(1,0),(0,1)\}$ con $K = \mathbb{C}$

B ortonormal

$\Phi(x,y) = \langle x,y\rangle = x_1\bar{y}_1 - i x_1 \bar{y}_2 + i x_2 \bar{y}_1 + 2x_2\bar{y}_2$

$V_1' = V_1 = (1,0)$

$V_2' = V_2 - \left(\frac{\langle V_2, V_1'\rangle}{\|V_1'\|}\right) \cdot V_1'$

$\sqrt{\langle V_1', V_1'\rangle} = \sqrt{\langle V_1, V_1\rangle} = \sqrt{1 + i.0 + i.0 + 2.0} = 1$

$(0,1) - \left(\frac{i}{1}\right) \cdot (1,0)$

$(0,1),(1,0) = i$

$V_2' = (-i, 1)$

$\boxed{B'_{\mathbb{C}^2} = \{(1,0),(-i,1)\}}$ ortogonal $\longrightarrow$ $\boxed{B'_{\mathbb{C}^2} = \{(1,0),(-i,1)\}}$ ortonormal

$\|(1,0)\| = \sqrt{1} = 1$

$\|(-i,1)\| = \sqrt{-i^2 - i.(-i).1 + i.1.i + 2.1.1} = \sqrt{1} = 1$

$-i^2 + i^2 + i^2 + 2$

3. requiero $B = \{v_1, v_2, \dots, v_n\}$ ortonormal $\Rightarrow$ $\langle v_i, v_j \rangle = 0$, $\langle v_i, v_i \rangle = 1$

i)

$$\langle X, Y \rangle = [X]_B^t \cdot A \cdot [Y]_B$$

$\mathbb{V} = \mathbb{R}^2$

$$\Phi = [\alpha_1 \ \ \alpha_2] \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \underset{1\times 2}{[\alpha_1 \ \ \alpha_2]} \cdot \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\Phi = (\alpha_1 \ \ \alpha_2) \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2$$

$$X = \alpha_1 (1,1) + \alpha_2 (2,-1) \qquad x_1 = \alpha_1 + 2\alpha_2 \qquad x_2 + 3\alpha_2 = x_1$$
$$Y = \beta_1 (1,1) + \beta_2 (2,-1) \qquad x_2 = \alpha_1 - \alpha_2 \qquad \alpha_1 = x_2 + \alpha_2$$

NOTA del ej. 2.

$(1,0)$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad \alpha_1 = x_1 - 2x_2$

$x_1 = \alpha_1 + \frac{2}{\sqrt{2}}\alpha_2 \quad \alpha_2 = \sqrt{2}\, x_2$

$x_2 = \frac{\alpha_2}{\sqrt{2}}$

y_1 $\quad = (x_1 - 2x_2) \cdot (y_1 - 2y_2) + (\sqrt{2} x_2)(\sqrt{2} y_2)$

y_2 $\quad \Phi = x_1 y_1 - 2x_2 y_1 - 2y_2 x_1 + \underbrace{4 x_2 y_2 + 2 x_2 y_2}_{+ 6 x_2 y_2}$

Haciendo el proceso al revés llego al Φ inicial

$$y_1 = \beta_1 + 2\beta_2 \qquad \alpha_2 = \frac{x_1 - x_2}{3}$$
$$y_2 = \beta_1 - \beta_2 \qquad \alpha_1 = x_2 + \frac{x_1}{3} - \frac{x_2}{3}$$

$$\beta_1 = y_2 + \beta_2 \qquad \alpha_1 = \frac{2}{3}x_2 + \frac{1}{3}x_1$$

$$y_1 = y_2 + 3\beta_2$$
$$\beta_2 = \frac{y_1 - y_2}{3}$$
$$\beta_1 = \frac{2}{3}y_2 + \frac{1}{3}y_1$$

$$\Phi = \langle X, Y \rangle = \left(\frac{2}{3}x_2 + \frac{1}{3}x_1\right) \cdot \left(\frac{2}{3}y_2 + \frac{1}{3}y_1\right) + \left(\frac{x_1 - x_2}{3}\right) \cdot \left(\frac{y_1 - y_2}{3}\right)$$

$$\Phi(x,y) = \frac{4}{9}x_2 y_2 + \frac{2}{9}x_1 y_2 + \frac{2}{9}x_2 y_1 + \frac{1}{9}x_1 y_1 + \frac{x_1 y_1}{9} - \frac{x_2 y_1}{9} - \frac{x_1 y_2}{9} + \frac{x_2 y_2}{9}$$

$$\boxed{\Phi(x,y) = \frac{5}{9}x_2 y_2 + \frac{1}{9}x_1 y_2 + \frac{1}{9}x_2 y_1 + \frac{2}{9}x_1 y_1}$$

ii)

$$\langle x, y \rangle = [X]_B^t \cdot A \cdot \overline{[Y]}_B$$

$\mathbb{V} = \mathbb{C}^2 \quad B = \{(1,i), (-1,i)\}$ Si es B ortonormal $\Rightarrow A = I$

$\therefore$

$$\langle x, y \rangle = X^t \cdot A \cdot \overline{Y} = X^t \cdot \overline{Y}$$
$$= (\alpha_1 \ \ \alpha_2) \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \end{pmatrix} = \alpha_1 \cdot \overline{\beta_1} + \alpha_2 \cdot \overline{\beta_2}$$

$$X = \alpha_1 (1,i) + \alpha_2 (-1,i) \longrightarrow$$
$$Y = \overline{\beta_1} (1,i) + \overline{\beta_2} (-1,i) \longrightarrow$$

$$\begin{cases} x_1 = \alpha_1 - \alpha_2 \\ x_2 = i \cdot \alpha_1 + i \cdot \alpha_2 \end{cases} \qquad \begin{cases} y_1 = \overline{\beta_1} - \overline{\beta_2} \\ y_2 = \overline{\beta_1} i + \overline{\beta_2} i \end{cases}$$

$$\alpha_1 = x_1 + \alpha_2$$
$$x_2 = x_1 i + \alpha_2 i + \alpha_2 i \qquad \alpha_1 = x_1 + \frac{x_2}{2i} - \frac{x_1}{2} = \frac{x_1}{2} - \frac{x_2 i}{2}$$
$$\frac{x_2 - x_1 i}{2i} = \alpha_2$$

$$\alpha_1 = \frac{1}{2}x_1 - \frac{1}{2}x_2 i \qquad \bar{\beta}_1 = \frac{1}{2}y_1 - \frac{1}{2}y_2 i$$

$$\alpha_2 = -\frac{1}{2}x_2 i - \frac{1}{2}x_1 \qquad \bar{\beta}_2 = -\frac{1}{2}y_2 i - \frac{1}{2}y_1$$

$$\Phi(x,y) = \langle x,y\rangle = \left(\frac{1}{2}x_1 - \frac{1}{2}x_2 i\right)\cdot\left(\frac{1}{2}y_1 - \frac{1}{2}y_2 i\right) + \left(-\frac{1}{2}x_2 i - \frac{1}{2}x_1\right)\cdot\left(-\frac{1}{2}y_2 i - \frac{1}{2}y_1\right)$$

$$\Phi = \frac{1}{4}x_1y_1 - i\frac{1}{4}x_2y_1 - i\frac{1}{4}x_1y_2 + i^2\frac{1}{4}x_2y_2 + i^2\frac{1}{4}x_2y_2 + i\frac{1}{4}x_1y_2 + i\frac{1}{4}x_2y_1 + \frac{1}{4}x_1y_1$$

$$\boxed{\langle x,y\rangle = \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2}$$ Producto interno solicitado

iii) $\mathbb{V} = \mathbb{R}^3 \qquad B = \{\overset{v_1}{(1,-1,1)}, \overset{v_2}{(1,1,0)}, \overset{v_3}{(0,1,1)}\}$

Necesito B ortonormal $\Rightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$

$$\langle x,y\rangle = \overset{1\times3}{[X]^t_B}\overset{3\times3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}[Y]_B$$

$$\langle x,y\rangle = (\alpha_1 \ \alpha_2 \ \alpha_3)\cdot\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

$X = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
$Y = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

$x_1 = \alpha_1 + \alpha_2 \qquad y_1 = \beta_1 + \beta_2$
$x_2 = -\alpha_1 + \alpha_2 + \alpha_3 \qquad y_2 = -\beta_1 + \beta_2 + \beta_3$
$x_3 = \alpha_1 + \alpha_3 \qquad y_3 = \beta_1 + \beta_3$

$$\left(\begin{array}{ccc|c|c} 1 & 1 & 0 & x_1 & y_1 \\ -1 & 1 & 1 & x_2 & y_2 \\ 1 & 0 & 1 & x_3 & y_3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|c} 1 & 1 & 0 & x_1 & y_1 \\ 0 & 2 & 1 & x_2+x_1 & y_2+y_1 \\ 0 & -1 & 1 & x_3-x_1 & y_3-y_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|c} 1 & 1 & 0 & x_1 & y_1 \\ 0 & 2 & 1 & x_2+x_1 & y_2+y_1 \\ 0 & 0 & 3 & 2x_3-2x_1+x_2+x_1 & 2y_3-2y_1+y_2+y_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|c} 0 & 6 & 0 & 3x_2+3x_1-\varphi & 3y_2+3y_1-\phi \\ 0 & 0 & 3 & 2x_3-x_1+x_2 & 2y_3-y_1+y_2 \end{array}\right)$$

$$\alpha_3 = \frac{2x_3}{3} - \frac{x_1}{3} + \frac{x_2}{3}$$

$$\alpha_2 = \frac{x_2}{2} + \frac{x_1}{2} - \frac{x_3}{3} + \frac{x_1}{3} - \frac{x_2}{6} - \frac{x_1}{6} = \frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3$$

$$\alpha_1 = \frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3$$

Para $\beta_1, \beta_2, \beta_3$ resultan las mismas expresiones pero con $x_i = y_i$ para $i = 1,2,3$

$$\langle X, Y \rangle = \left(\frac{X_1}{3} - \frac{X_2}{3} + \frac{X_3}{3}\right) \cdot \left(\frac{Y_1}{3} - \frac{Y_2}{3} + \frac{Y_3}{3}\right) + \left(\frac{2X_1}{3} + \frac{X_2}{3} - \frac{X_3}{3}\right) \cdot \left(\frac{2Y_1}{3} + \frac{Y_2}{3} - \frac{Y_3}{3}\right) + \left(-\frac{X_1}{3} + \frac{X_2}{3} + \frac{2X_3}{3}\right) \cdot \left(-\frac{Y_1}{3} + \frac{Y_2}{3} + \frac{2Y_3}{3}\right)$$

multiplicación

4. $\phi(x,y) = \langle x,y \rangle =$

$$a x_1 y_1 + b x_1 y_2 + b x_2 y_1 + b x_2 y_2 + (1+b) x_3 y_3$$

producto interno en $\mathbb{R}^3$

$$\langle \lambda_1 \overset{x_1,x_2,x_3}{X} + \lambda_2 \overset{z_1,z_2,z_3}{Z}, \overset{y_1,y_2,y_3}{Y} \rangle = \langle (\lambda_1 x_1 + \lambda_2 z_1, \lambda_1 x_2 + \lambda_2 z_2, \lambda_1 x_3 + \lambda_2 z_3), (y_1, y_2, y_3) \rangle$$

$$= a(\lambda_1 x_1 + \lambda_2 z_1) y_1 + b(\lambda_1 x_1 + \lambda_2 z_1) y_2 + b(\lambda_1 x_2 + \lambda_2 z_2) y_1 + b(\lambda_1 x_2 + \lambda_2 z_2) y_2$$
$$+ (1+b)(\lambda_1 x_3 + \lambda_2 z_3) y_3$$

$$= \lambda_1 (a x_1 y_1 + b x_1 y_2 + b x_2 y_1 + b x_2 y_2 + (1+b) x_3 y_3) + \lambda_2 (a z_1 y_1 + b z_1 y_2 + b z_2 y_1 + b z_2 y_2$$
$$+ (1+b) . z_3 y_3) = \lambda_1 \langle X, Y \rangle + \lambda_2 \langle Z, Y \rangle$$

$$\phi(x,x) = \langle x,x \rangle = a x_1^2 + b x_1 x_2 + b x_2 x_1 + b x_2^2 + (1+b) x_3^2$$
$$= a x_1^2 + 2b x_1 x_2 + b x_2^2 + (1+b) x_3^2$$
$$x_1^2 + 2\frac{b}{a} x_1 x_2 + \frac{b}{a} x_2^2 + \left(\frac{b+1}{a}\right) x_3^2$$

$$\underbrace{x_1^2 + 2\frac{b}{a} x_1 x_2 + \frac{b^2}{a^2} x_2^2} - \frac{b^2}{a^2} x_2^2 + \frac{b}{a} x_2^2 + \frac{(b+1)}{a} x_3^2$$

$$\left(x_1 + \frac{b}{a} x_2\right)^2 + x_2^2 \left(\frac{b}{a} - \frac{b^2}{a^2}\right) + x_3^2 \left(\frac{b+1}{a}\right)$$

$$x_2^2 \left(\frac{ab - b^2}{a^2}\right)$$

$$\left(x_1 + \frac{b}{a} x_2\right)^2 + x_2^2 \left[\frac{a^2 + (a-b)^2}{a^2}\right] + x_3^2 \left(\frac{b+1}{a}\right) > 0$$

muy complicado $\Rightarrow$ se usarán matrices

$e_1 = (1,0,0) \quad e_2 = (0,1,0) \quad e_3 = (0,0,1)$

$$A = \begin{pmatrix} \langle e_1,e_1 \rangle & \langle e_1,e_2 \rangle & \langle e_1,e_3 \rangle \\ \langle e_2,e_1 \rangle & \langle e_2,e_2 \rangle & \langle e_2,e_3 \rangle \\ \langle e_3,e_1 \rangle & \langle e_3,e_2 \rangle & \langle e_3,e_3 \rangle \end{pmatrix} \qquad [x_1 \ \ x_2] . A . \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & 0 \\ b & b & 0 \\ 0 & 0 & 1+b \end{pmatrix} \quad \xrightarrow{x\,a/b + f_1} \begin{pmatrix} a & b & 0 \\ a & a & 0 \\ 0 & 0 & 1+b \end{pmatrix} \quad \begin{pmatrix} a & b & 0 \\ 0 & a-b & 0 \\ 0 & 0 & 1+b \end{pmatrix} \rightarrow \begin{pmatrix} \frac{a}{b}(a-b) & a-b & 0 \\ 0 & a-b & 0 \\ 0 & 0 & 1+b \end{pmatrix}$$

es Real y simétrica

$$\begin{cases} a \neq 0 \\ b \neq 0 \end{cases} \qquad a \neq b$$

$$A = \begin{pmatrix} \frac{a^2 - ba}{b} & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & 1+b \end{pmatrix}$$

es definida positiva $\Leftrightarrow$ $a > 0$; $a - b > 0 \quad a > b$; $b + 1 > 0 \quad b > -1$

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & 1+b \end{pmatrix}$$

$a > b > -1$

dos posibilidades $\begin{cases} \boxed{a > b > 0 > -1} \\ \boxed{a > 0 > b > -1} \end{cases}$

5. 1) $\langle A,B \rangle = \text{traza}\,(A.B^*)$ $\qquad AB^* \in K^{n\times n}$

$tr(AB^*) = tr\,(AB^*)^t$

si A es diagonalizable $\Rightarrow$

$tr\,(A) = tr\,(C^{-1}.J.C)$

Sean: $e_1 = \{\overbrace{1,0,0,\ldots,0}^{n^2}\}$
$e_2 = (0,1,0,\ldots,0)$ base de $K^{n\times n}$
$\ldots$
$e_{n^2} = (0,0,\ldots,0,1)$

son ortonormales $\Rightarrow$ $\langle e_i, e_j \rangle = \begin{cases} 0 & \text{si } i \neq j \\ 1 & \text{si } i = j \end{cases}$

ii) $\langle , \rangle : C[0,1] \times C[0,1] \to \mathbb{R}$, $\langle f,g \rangle = \int_0^1 f(x).g(x)\,dx$

$\langle f,f \rangle = \int_0^1 f^2(x).dx > 0 \quad \wedge \quad \int_0^1 f^2(x).dx = 0 \Leftrightarrow f(x)=0 \quad \forall x \in [0,1]$

pues $f^2(x) \geq 0 \quad \forall x$

$\langle f,g \rangle = \int_0^1 f(x).g(x).dx = \int_0^1 g(x).f(x).dx = \langle g,f \rangle$

$\langle f+h,g \rangle = \int_0^1 [f(x)+h(x)].g(x).dx = \int_0^1 f(x).g(x).dx + \int_0^1 h(x).g(x).dx$
$= \langle f,g \rangle + \langle h,g \rangle$

$\langle \lambda f,g \rangle = \int_0^1 \lambda f(x).g(x).dx = \lambda \int_0^1 f(x).g(x)\,dx = \lambda \langle f,g \rangle$

$\Rightarrow$ $\boxed{\langle f,g \rangle \text{ es un producto interno}}$

6.

$\mathbb{R}_n[X]$ polinomios de coeficientes reales en X

matriz en la base $\quad B = \{1, X, X^2, \dots, X^n\}$

bose infinita

$$\langle 1,1\rangle = \int_0^1 1\,dx = 1 \qquad \langle X,1\rangle = \int_0^1 X\,dx = 1/2$$

$$\langle 1,X\rangle = \int_0^1 X\,dx = \frac{1}{2} \qquad \langle X,X\rangle = \int_0^1 X^2\,dx = 1/3$$

$$\langle 1,X^2\rangle = \int_0^1 X^2\,dx = \frac{1}{3} \qquad \langle X,X^2\rangle = \int_0^1 X^3\,dx = 1/4$$

$$\dots \qquad \dots$$

$$\langle X^2,1\rangle = \int_0^1 X^2\,dx = 1/3$$

$$\langle X^2,X\rangle = \int_0^1 X^3\,dx = 1/4$$

$$\boxed{A = \begin{pmatrix} 1 & 1/2 & 1/3 & \dots & 1/n \\ 1/2 & 1/3 & 1/4 & & 1/(n+1) \\ 1/3 & 1/4 & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 1/n & 1/(n+1) & \dots & \dots & 1/(2n+1) \end{pmatrix}}$$

$$\langle X^n,X^n\rangle = \int_0^1 X^{2n}\,dx = \frac{1}{(2n+1)}$$

7. i). $\mathbb{V}=\mathbb{R}^3 \quad S_1=\{\bar{X}\in\mathbb{R}^3 : 2X_1-X_2=0\}$

$$\boxed{S_1^{\perp}=\langle(2,-1,0)\rangle}$$

ii) $\mathbb{V}=\mathbb{R}^3 \quad S_2=\langle(1,2,1)\rangle$

a)

$$\boxed{S_2^{\perp}=\langle(-2,1,0),(-1,0,1)\rangle}$$

$$(1,2,1).(x_1,x_2,x_3)=0$$
$$x_1+2x_2+x_3=0$$
$$x_1=-x_3-2x_2$$

$$S^{\perp}=\{(-2x_2-x_3,\ x_2,x_3)\}$$

$$S^{\perp}=\langle(-2,1,0),(-1,0,1)\rangle$$

b) $\langle V_1,V_2\rangle = x_1y_1+2x_2y_2+x_3y_3-x_1y_2-x_2y_1$

$$\langle(1,2,1),(x_1,x_2,x_3)\rangle=0$$

$$x_1+4x_2+x_3-x_2-2x_1=-x_1+3x_2+x_3=0$$

$$x_1=3x_2+x_3$$

$$(x_1,x_2,x_3)=(3x_2+x_3,\ x_2,x_3)$$

$$\boxed{S^{\perp}=\langle(3,1,0),(1,0,1)\rangle}$$

iii) $\mathbb{V}=\mathbb{C}^3 \quad S_3=\langle(i,1,1),(-1,0,i)\rangle$

$$\left.\begin{aligned}(x,y,z).(i,1,1)&=-xi+y+z=0\\(x,y,z).(-1,0,i)&=-x-zi=0\end{aligned}\right\}\ \text{sistema}$$

$$x=-zi \ \to\ zi^2+y+z=y=0\ \to$$

$$S_3^{\perp}=(-zi,0,z)$$

$$\boxed{S_3^{\perp}=\langle(-i,0,1)\rangle}$$

$(-i,0,1).(i,1,1)=i^2+1=0$
$(-i,0,1).(-1,0,i)=i-i$

iv) $\mathbb{V}=\mathbb{C}^4 \quad S_4=\left\{\bar{x}\in\mathbb{C}^4 : \begin{cases}x_1+2ix_2-x_3+(1+i)x_4=0\\x_2+(2-i)x_3+x_4=0\end{cases}\right\}$

$$\langle x,y\rangle = x_1\bar{y}_1+2x_2\bar{y}_2+x_3\bar{y}_3+3x_4\bar{y}_4$$

$$S_4=\left\langle\left[(-2+i),-\left(\tfrac{1}{2}+\tfrac{3}{2}i\right),1,0\right],\left[-1,-\tfrac{1}{2},0,1\right]\right\rangle \qquad \left(\begin{array}{cccc|c}1&2i&-1&1+i&0\\0&1&2-i&1&0\end{array}\right)$$

$$\begin{cases}(x_1,x_2,x_3,x_4).(-2+i,-1/2-3/2i,1,0)=0 & (1)\\(x_1,x_2,x_3,x_4).(-1,-1/2,0,1)=0 & (2)\end{cases}$$

$$x_1=-x_4-(2-i)x_3$$

$$0=-x_4-(2-i)x_3+2ix_2-x_3+(1+i)x_4$$
$$x_2=+\frac{(-3+i)}{-2i}x_3+\frac{i}{-2i}x_4$$

$\frac{1}{i}=-i$

$$x_2=-\frac{1}{2}x_4+\left(-\frac{1}{2}-\frac{3}{2}i\right)x_3$$

Con (1) y (2) formaremos el sistema según la definición de $\langle x,y\rangle$

$$\bar{X}_{\in S_4}=\left((-2+i)x_3-x_4,-\frac{1}{2}x_4-\left(\frac{1}{2}+\frac{3}{2}i\right)x_3,\ x_3,\ x_4\right)$$

$$\begin{cases}(2-i)x_1+(-1+3i)x_2+x_3=0\\ -x_1-2x_2\cdot\frac{1}{2}+3x_4=0\end{cases}$$

$$\begin{cases}(2-i)x_1+(-1+3i)x_2+x_3=0\\ -x_1-x_2+3x_4=0\end{cases}$$

$$x_1=-x_2+3x_4 \longrightarrow \begin{array}{l}(-2-i)(-x_2+3x_4)+(-1+3i)x_2+x_3=0\\ (2+i)x_2+(-6-3i)x_4+(-1+3i)x_2+x_3=0\\ x_3=-(1+4i)x_2-(-6-3i)x_4\end{array}$$

$$S^{\perp}=\langle(-x_2+3x_4,\ x_2,\ (-1-4i)x_2+(6+3i)x_4,\ x_4)\rangle$$

$$\boxed{S^{\perp}=\langle[(-1,1,(-1-4i),0),(3,0,(6+3i),1)]\rangle}$$

v) $V=\mathbb{R}^4$ $\quad S_5=\langle(1,1,0,-1),(-1,1,1,0),(2,-1,1,1)\rangle$

$$\begin{cases}(x_1,x_2,x_3,x_4)\cdot(1,1,0,-1)=0\\ (x_1,x_2,x_3,x_4)\cdot(-1,1,1,0)=0\\ (x_1,x_2,x_3,x_4)\cdot(2,-1,1,1)=0\end{cases}$$

$$\begin{array}{r}x_1+x_2-x_4=0\\ -x_1+x_2+x_3=0\\ 2x_1-x_2+x_3+x_4=0\end{array}$$

$$\left(\begin{array}{cccc|c}1&1&0&-1&0\\-1&1&1&0&0\\2&-1&1&1&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&1&0&-1&0\\0&2&1&-1&0\\0&-3&1&3&0\end{array}\right)$$

$$\left(\begin{array}{cccc|c}1&1&0&-1&0\\0&1&1/2&-1/2&0\\0&-1&1/3&1&0\end{array}\right)\left(\begin{array}{cccc|c}1&1&0&-1&0\\0&1&1/2&-1/2&0\\0&0&5/6&-1/2&0\end{array}\right)$$

$$\frac{5}{6}x_3=\frac{1}{2}x_4 \qquad x_3=\frac{3}{5}x_4$$

$$x_2=\frac{1}{2}x_4-\frac{1}{2}x_3$$

$$x_2=\frac{x_4}{2}-\frac{1}{2}\frac{3}{5}x_4=\frac{1}{5}x_4$$

$$x_1=x_4-x_2$$

$$x_1=x_4-\frac{x_4}{5}=\frac{4}{5}x_4$$

$$S_5^{\perp}=\langle(4/5x_4,\ 1/5x_4,\ 3/5x_4,\ x_4)\rangle$$

$$\boxed{S_5^{\perp}=\langle(\tfrac{4}{5},\tfrac{1}{5},\tfrac{3}{5},1)\rangle}$$

8.

i) (i) $\{2x_1-x_2=0 \qquad x_2=2x_1 \qquad S_1=\langle(1,2,0),(0,0,1)\rangle$

$$B_{S_1}=\{(1,2,0),(0,0,1)\}$$

ortogonalizar:

$$V_1'=(1,2,0)$$

$$\|V_1\|=\sqrt{1+4}=\sqrt{5} \qquad V_2'=(0,0,1)-\left(\frac{(0,0,1)\cdot(1,2,0)}{\sqrt{5}}\right)(1,2,0)=(0,0,1)-\frac{0}{\sqrt{5}}(1,2,0)=(0,0,1)$$

ortonormalizar: $U_1=\frac{(1,2,0)}{\sqrt{5}}$ $\quad U_2=(0,0,1)$

$$\boxed{B'_{S_1}=\{(1/\sqrt{5},2/\sqrt{5},0),(0,0,1)\}}$$

(ii) a) 1 vector es ortogonal $\Rightarrow$ ortonormalizamos

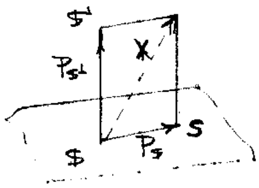
$$U=\frac{(1,2,1)}{\sqrt{6}}=\left(1/\sqrt{6},2/\sqrt{6},1/\sqrt{6}\right)$$

$$\boxed{B'_{S_2}=\{(1/\sqrt{6},2/\sqrt{6},1/\sqrt{6})\}}$$

11) ⊛ $S_1 = \{X \in \mathbb{R}^3 : 2x_1 - x_2 = 0\}$ $\quad x_2 = 2x_1$

$S_1^\perp = \langle (2,-1,0) \rangle \qquad S_1 = \langle (1,2,0),(0,0,1) \rangle$

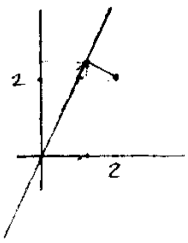
$P_S(x)$



$$\langle \overbrace{X - \underbrace{P_S(x)}_{Y}}^{P_{S^\perp}(x)}, s \rangle = 0$$

$$\left[\begin{array}{l} \langle (X-Y),(1,2,0) \rangle = 0 \\ \langle (X-Y),(0,0,1) \rangle = 0 \\ 2y_1 - y_2 \quad = 0 \end{array}\right.$$

$\langle X,(1,2,0)\rangle = \langle Y$



$$\begin{cases} x_1 + 2x_2 = y_1 + 2y_2 \\ x_3 = y_3 \\ 0 = 2y_1 - y_2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & x_1+2x_2 \\ 0 & 0 & 1 & x_3 \\ 2 & -1 & 0 & 0 \end{array}\right)$$

$y_3 = x_3$

$y_1 = x_1 + 2x_2 - 4y_1$
$y_1 = x_1/5 + 2/5\, x_2$
$y_2 = 2y_1$

$$\boxed{P_{S_1}(x) = \left(\frac{x_1}{5} + \frac{2}{5}x_2, \frac{2}{5}x_1 + \frac{4}{5}x_2, x_3\right)}$$

⊛ $S_2 = \langle (1,2,1) \rangle \qquad \mathbb{V} = \mathbb{R}^3$

a) $\langle X, (1,2,1) \rangle = 0 \rightarrow$ $\qquad x_1 + 2x_2 + x_3 = 0$

$S^\perp = \{v \in \mathbb{V} : \langle v, s \rangle = 0 \ \forall s \in S\}$

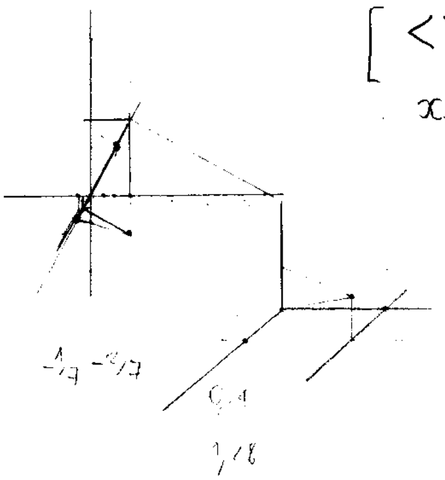
$s^\perp \in S_2^\perp \ (x_1, x_2, -x_1 - 2x_2)$

$S_2^\perp = \langle (1,0,-1),(0,1,-2) \rangle$

$\langle X - Y, \langle S \rangle \rangle = 0 \qquad Y, 1$

$$\left[\ \langle X,(1,2,1)\rangle = \langle Y,(1,2,1)\rangle \right. \qquad \begin{array}{l}\langle Y,(1,0,-1)\rangle = 0 \\ \langle Y,(0,1,-2)\rangle = 0\end{array}$$

$$\begin{array}{c} x_1 + 2x_2 + x_3 = y_1 + 2y_2 + y_3 \\ 0 = y_1 - y_3 \\ 0 = y_2 - 2y_3 \end{array}$$



$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & x_1+2x_2+x_3 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$y_1 = x_1 + 2x_2 + x_3 - y_3 - 2y_2$

$y_2 = 2y_3$
$y_1 = y_3$

$\downarrow$

$6y_3 = x_1 + 2x_2 + x_3$

$-2.\frac{y_2}{2} - 2y_2$

$-2.(2y_3)$

$P_{S_2}(x) = (y_1, y_2, y_3)$ con $\rightarrow$ $\boxed{\begin{array}{l} y_1 = y_3 = \frac{1}{6}(x_1 + 2x_2 + x_3) \\ y_2 = \frac{1}{3}(x_1 + 2x_2 + x_3) \end{array}}$

10.

i) $f:\mathbb{R}^2\to\mathbb{R}^2 \quad f(x_1,x_2)=(3x_1+x_2,\,-x_1+x_2)$

$$M_E f=\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}$$

es ortonormal

$$M_E f^*=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix} \Rightarrow \boxed{f^*(x,y)=(3x_1-x_2,\,x_1+x_2)}$$

$f^*(1,0)=(3,1)$
$f^*(0,1)=(-1,1)$

$f(1,1)=(4,0) \quad \langle f(1,1),(2,2)\rangle=\langle(1,1),f^*(2,2)\rangle$

$f^*(2,2)=(4,4) \quad \langle(4,0),(2,2)\rangle=\langle(1,1),(4,4)\rangle$

$8 = 8$

ii) $f:\mathbb{C}^3\to\mathbb{C}^3$

$$M_E f=\begin{pmatrix}2 & (1-i) & 0\\ 0 & 1 & 3+2i\\ 1 & i & 1\end{pmatrix}$$

$$M_E f^*=\begin{pmatrix}2 & 0 & 1\\ 1+i & 1 & -i\\ 0 & 3-2i & 1\end{pmatrix}$$

$$\boxed{f^*(x_1,x_2,x_3)=(2x_1+x_3,\,(1+i)x_1+x_2-ix_3,\,(3-2i)x_2+x_3)}$$

iii) $B=\{(1,2,-1),(1,0,0),(0,1,1)\} \quad f:\mathbb{R}^3\to\mathbb{R}^3$

$$M_{BB}f=\begin{pmatrix}1 & 0 & 1\\ 2 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} \qquad C_{BE}=M_{BE}I=\begin{pmatrix}1 & 1 & 0\\ 2 & 0 & 1\\ -1 & 0 & 1\end{pmatrix}\Rightarrow$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 2 & 0 & 1 & 0 & 1 & 0\\ -1 & 0 & 1 & 0 & 0 & 1\end{array}\right) \left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & -2 & 1 & -2 & 1 & 0\\ 0 & 1 & 1 & 1 & 0 & 1\end{array}\right) \left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & -3 & 0 & -3 & 1 & -1\\ 0 & 0 & 3 & 0 & 1 & 2\end{array}\right)$$

$$\left(\begin{array}{ccc|ccc}3 & 0 & 0 & 0 & 1 & -1\\ 0 & -3 & 0 & -3 & 1 & -1\\ 0 & 0 & 3 & 0 & 1 & 2\end{array}\right) \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 1/3 & -1/3\\ 0 & 1 & 0 & 1 & -1/3 & 1/3\\ 0 & 0 & 1 & 0 & 1/3 & 2/3\end{array}\right)$$

$$M_{EE}f=C_{BE}\cdot M_{BB}f\cdot C_{EB}$$

$$=\begin{pmatrix}1 & 1 & 0\\ 2 & 0 & 1\\ -1 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 1\\ 2 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 1/3 & -1/3\\ 1 & -1/3 & 1/3\\ 0 & 1/3 & 2/3\end{pmatrix}$$

$$M_E f=\begin{pmatrix}0 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & 0\end{pmatrix}$$

$$f(x_1,x_2,x_3)=(x_2-x_3,\,x_1+x_2+x_3,\,x_1-x_2)$$

$$M_E f^*=\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & -1\\ -1 & 1 & 0\end{pmatrix}$$

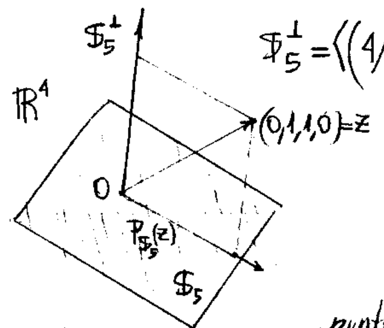
$$\boxed{f^*(x_1,x_2,x_3)=(x_2+x_3,\,x_1+x_2-x_3,\,-x_1+x_2)}$$

iii) $\mathbb{S}_5 = \langle (1,1,0,-1),(-1,1,1,0),(2,-1,1,1) \rangle \qquad \mathbb{V} = \mathbb{R}^4$

$$\begin{cases} \langle X,(1,1,0,-1)\rangle = 0 \\ \langle X,(-1,1,1,0)\rangle = 0 \\ \langle X,(2,-1,1,1)\rangle = 0 \end{cases}$$

$$\begin{aligned} x_1 + x_2 - x_4 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 + x_4 &= 0 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & -3 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & -2 & 2/3 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 5/3 & 1 & 0 \end{array}\right)$$



$\mathbb{S}_5^\perp = \langle (4/5, 1/5, 3/5, 1) \rangle$

$$\begin{aligned} x_1 &= x_4 - x_2 \\ x_1 &= x_4 - \tfrac{1}{5}x_4 \\ x_1 &= \tfrac{4}{5}x_4 \end{aligned} \qquad \begin{aligned} x_3 &= \tfrac{3}{5}x_4 \\ x_2 &= \tfrac{x_4}{2} - \tfrac{3}{5.2}x_4 = \tfrac{1}{5}x_4 \end{aligned}$$

punto mas cercano.

$P_{\mathbb{S}_5}(0,1,1,0)$

$$z = P_{\mathbb{S}}(z) + P_{\mathbb{S}^\perp}(z)$$

$$\langle \underbrace{z - P_{\mathbb{S}}(z)}_{\equiv P_{\mathbb{S}^\perp}(z)}, \overset{\in \mathbb{S}_5}{s} \rangle = 0$$

$$\left[\begin{array}{l} \langle z-y,(1,1,0,-1)\rangle = 0 \\ \langle z-y,(-1,1,1,0)\rangle = 0 \\ \langle z-y,(2,-1,1,1)\rangle = 0 \\ \langle y, \langle \mathbb{S}_5^\perp \rangle \rangle = 0 \end{array}\right.$$

$$\begin{aligned} z_1 + z_2 - z_4 &= y_1 + y_2 - y_4 \\ -z_1 + z_2 + z_3 &= -y_1 + y_2 + y_3 \\ 2z_1 - z_2 + z_3 + z_4 &= 2y_1 - y_2 + y_3 + y_4 \\ 0 &= \tfrac{4}{5}y_1 + \tfrac{y_2}{5} + \tfrac{3}{5}y_3 + y_4 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 1 & 0 \\ 4/5 & 1/5 & 3/5 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & -1 & 3 \\ 0 & -3 & 1 & 3 & -2 \\ 4 & 1 & 3 & 5 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & -1 & 3 \\ 0 & -3 & 1 & 3 & -2 \\ 0 & -3 & 3 & 9 & -4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & -1 & 3 \\ 0 & -1 & 1/3 & 1 & -2/3 \\ 0 & 0 & 2 & 6 & -2 \end{array}\right) \approx \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 2 & 0 & 0 & 3/2 \\ 0 & 0 & 5/2 & 0 & 15/4 \\ 0 & 0 & 0 & 24/25 & -5/4 \end{array}\right)$$

$$\boxed{P_{\mathbb{S}_5}(0,1,1,0) = \left(6, \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{125}{96}\right)}$$

distancia entre ambos puntos

$$d(P_{\mathbb{S}_5}(z), z) = \| P_{\mathbb{S}_5}(z) - z \| = \sqrt{(6-0)^2 + (3/4-1)^2 + (3/2-1)^2 + \left(\tfrac{125}{96}\right)^2}$$

$$d(P_{\mathbb{S}_5}(z), z) = \boxed{6{,}165}$$