PRACTICA 4

1. (a) $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \frac{1}{2}|x|$ si $\frac{|x|}{2} < 1$ $|x|<2$

conv. abs. en $\boxed{(-2,2)}$

si $x=2 \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1$ diverge

si $x=-2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} -1$ diverge

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n . x^n}{n^2}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \frac{n^2 . |x|}{(n^2+2n+1)} = |x|$ $|x|<1$

conv. abs. en $(-1,1)$

si $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\frac{1}{n^2}\to 0$ $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ decreciente $\frac{1}{n^2}>0$

conv. x Leibnitz

si $x=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n . (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converge

con. abs en $[-1,1]$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n+\sqrt{n}}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1+\sqrt{n+1}} \cdot \frac{n+\sqrt{n}}{x^n} \right| = \lim_{n\to\infty} \frac{|x| . (n+\sqrt{n})}{n+1+\sqrt{n+1}} =$

$\lim_{n\to\infty} \frac{|x|\left(\frac{n+\sqrt{n}}{n}\right)}{\frac{n+1+\sqrt{n+1}}{n}} = \lim_{n\to\infty} \frac{|x| . (1+1/\sqrt{n})}{1+\frac{1}{n}+\sqrt{\frac{1}{n}+\frac{1}{n^2}}} = |x|$ $|x|<1$

conv. abs. en $(-1,1)$

si $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverge $n+\sqrt{n} < 2n$

$\frac{1}{n+\sqrt{n}} > \frac{1}{2n}$

$\sum \frac{1}{n+\sqrt{n}} > \sum \frac{1}{2n}$

diverge $\Leftarrow$ diverge

$\boxed{\text{converge en } [-1,1)}$

si $x=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$

$\frac{1}{n+\sqrt{n}} \to 0$ $n+\sqrt{n} < n+1+\sqrt{n+1}$

$\frac{1}{n+\sqrt{n}} > \frac{1}{n+1+\sqrt{n+1}}$ decreciente

$\frac{1}{n+\sqrt{n}} > 0 \ \forall n \in \mathbb{N}$ $\Rightarrow$ conv. x Leibnitz

(d) $\sum_{n=1}^{\infty} 2^n . \text{sen}\left(\frac{x}{3^n}\right)$ $\lim_{n\to\infty} \left| 2^{n+1} \text{sen}\left(\frac{x}{3^{n+1}}\right) \cdot \frac{1}{2^n} \cdot \frac{1}{\text{sen}\left(\frac{x}{3^n}\right)} \right| =$

$\lim_{n\to\infty} \left| \frac{2 . \text{sen}\left(\frac{x}{3^{n+1}}\right) \frac{x}{3^n} \frac{x}{3^{n+1}}}{\text{sen}\left(\frac{x}{3^n}\right) \frac{x}{3^n} \frac{x}{3^{n+1}}} \right| = \frac{2}{3} < 1 \Rightarrow$ conv. $\forall$ $x \in \mathbb{R}$

$\frac{x}{3^{n+1}}$

$\boxed{R = \pm\infty}$

3. $\sum_{n=1}^{\infty} n . x^n$ si $|x|<1$ converge $\lim_{n\to\infty} \left| \frac{(n+1) . x^{n+1}}{n . x^n} \right| = |x|<1$ conv $(-1,1)$

$x=1 \Rightarrow \sum n$ div

$x=-1 \Rightarrow \sum n(-1)^n$

$(-1)^n n \not\to 0$ div

$= x + 2x^2 + 3x^3 + 4x^4 + \dots + n . x^n$

$\sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n$ Converge si $|x|<1$

$= \sum_{n=0}^{\infty} n . x^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{-(-1)}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \dots + n . x^{n-1}$

$\Rightarrow x . \sum_{n=0}^{\infty} n . x^{n-1} = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots + n . x^n = \sum_{n=1}^{\infty} n . x^n$

$\Rightarrow \sum_{n=1}^{\infty} n . x^n = \boxed{\frac{x}{(1-x)^2}}$

2. (a) $f(x) = \tan(x)$

$f'(x) = \tan^2(x) + 1 \qquad = 0 + x + 0 + \frac{2.x}{6} = x + \frac{1}{3}x$

$f''(x) = 2.\tan^3(x) + 2\tan(x)$

$f'''(x) = 6\tan^4(x) + 8\tan^2(x) + 2$

(b) $f(x) = e^{\cos x} \qquad = e + 0 + \frac{-e}{2}x^2 + 0 = e - \frac{e}{2}x^2$

$f'(x) = -\operatorname{sen}(x).e^{\cos(x)}$

$f''(x) = -\cos(x).e^{\cos(x)} + \operatorname{sen}^2(x).e^{\cos(x)}$

$f'''(x) = 3\operatorname{sen}(x).\cos(x).e^{\cos(x)} - \operatorname{sen}^3(x).e^{\cos(x)} + \operatorname{sen}(x).e^{\cos(x)}$

(c) $f(x) = \operatorname{Ln}(1+e^x)$

$f'(x) = \frac{e^x}{1+e^x} \qquad = (\operatorname{Ln} 2).x^0 + \frac{1}{2}.x + \frac{1}{4.2}.x^2 + \frac{0}{6}.x^3$

$f''(x) = \frac{e^x}{(1+e^x)^2} \qquad \boxed{\operatorname{Ln} 2 + \frac{x}{2} + \frac{x^2}{8} + 0}$

$f'''(x) = \frac{-e^x(e^x-1)}{(1+e^x)^3}$

(d) $f(x) = (1+x)^x = \qquad 1 + \sum_{n=1}^{+\infty} \frac{x(x-1)\dots(x-n+1)}{n!}x^n$

$f'(x) = (1+x)^x.\left[\operatorname{Ln}(1+x) + \frac{x}{1+x}\right]$

$x^4 - x^3 \qquad (x^2-x)(x-2)(x)^3$

$\operatorname{Ln}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad 1 + x.x + \frac{(x^2-x).x^2}{2!} + \frac{x(x-1)(x-2).x^3}{3!}$

$x.\operatorname{Ln}(1+x) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4}$

$1 + 0.x + \frac{2.x^2}{2} - \frac{3.x^3}{6} = \boxed{1 + x^2 - \frac{x^3}{2}}$

$\operatorname{Ln}(1+x)^x =$

4. (a) $\frac{1}{1-x} = \underbrace{\sum_{n=0}^{+\infty} x^n}_{} = \boxed{1 + x + x^2 + \dots + x^n}$

serie geométrica que converge a $\frac{1}{1-x}$ con $|x|<1$ — converge en $(-1,1)$

(b) $\sqrt{1+x} =$ con $\qquad (1+x)^{1/2}$

$f(x) = \sqrt{1+x} \qquad = \sqrt{1} + \frac{1}{2}x + -\frac{1}{4}x^2 + \frac{3}{8}x^3$

$f'(x) = \frac{1}{2\sqrt{1+x}}$

$\sum_{n=0}^{+\infty} (-1)^{n+1}.x^n = 1 + \frac{1}{2}x - \frac{1}{4.2}x^2 + \frac{3}{8.6}x^3 - \frac{15}{16.24}x^4$

$f''(x) = \frac{-1}{4\sqrt{(1+x)^3}}$

$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5.x^4}{128} \qquad \frac{3}{48}$

$f'''(x) = \frac{3}{8\sqrt{(1+x)^5}}$

$f''''(x) = \frac{-15}{16(1+x)^{7/2}} \qquad \operatorname{Ln}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n.\frac{x^{n+1}}{n+1} + \dots$

$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^4}{24}$

$1 + \sum_{n=1}^{+\infty} \frac{m(m-1)\dots(m-n+1)}{n!}.x^n = (1+x)^m \qquad = 1 + \left| \frac{1}{2.1!}x - \frac{1.1x^2}{4.2!} + \frac{1.3\ x^3}{8.3!} - \frac{1.3.5\ x^4}{16.4!} \right.$

1ro $\quad$ 2 $\quad$ 3 $\quad$ 4

$\sum_{n=1}^{+\infty} (-1)^{n+1}\left(\frac{1}{2}\right)^n.\frac{x^n}{n!}.\underbrace{1.3.5\dots(2n-1)}_{(n-1)\text{ terminos}}$

$$\sum_{n=1}^{+\infty} (-1)^{n+1}\cdot\left(\frac{1}{2}\right)^n\cdot\frac{x^n}{n!}\cdot\underbrace{1.3.5.\dots(2n-1)}_{(n-1)\text{ términos}}$$

$$(-1)^{n+1}\cdot\frac{x^n}{n!}\cdot\left[\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}-1\right)\cdot\dots\cdot\underbrace{\left(\frac{1}{2}-n+1\right)}_{\text{será el último término}}\right]$$

(i) $\left(\frac{1}{1+x}\right)=\left(\frac{1}{1-(-x)}\right)= 1-x+x^2-x^3+x^4-\dots-x^n$ si $|x|<1$

Convergencia de la serie geométrica

$$\left(\frac{1}{1+x}\right)'=\frac{0(1+x)-1(+1)}{(1+x)^2}=\frac{-1}{(1+x)^2}=0-1+2x-3x^2+4x^3-\dots-n.x^{n-1}$$

$$\frac{1}{(1+x)^2}=\frac{(-1)^2}{(1+x)^2}=(-1)\frac{-1}{(1+x)^2}=1-2x+3x^2-4x^3+\quad+n.x^{n-1}$$

$$\Rightarrow \boxed{\sum_{n=1}^{+\infty}(-1)^{n+1}n.x^{n-1}}=\frac{1}{(1+x)^2}$$

(d) $\frac{1}{1+x^2}=\frac{1}{1-\underbrace{(-x^2)}_{z}}=1-x^2+x^4-x^6+\dots+(-x^2)^n$ si $|x|<1$

$$1+z+z^2+z^3+\dots+z^n$$

$$\sum_{n=0}^{+\infty} z^n=\sum_{n=0}^{+\infty}(-x^2)^n=\boxed{\sum_{n=0}^{+\infty}(-1)^n.(x^2)^n}=\frac{1}{1+x^2}$$ converge en $|x^2|<1$, $|x|<1$

(c) $\frac{1}{10+x}=f(x)=\frac{1}{10\left(1+\frac{x}{10}\right)}=\frac{1}{10}\cdot\frac{1}{\left(1+\frac{x}{10}\right)}$

$$\frac{1}{10}\cdot\frac{1}{1-\left(-\frac{x}{10}\right)}=\frac{1}{10}\left(1-\frac{x}{10}+\frac{x^2}{10^2}-\frac{x^3}{10^3}+\dots-\left(\frac{x}{10}\right)^n\right)$$ $\left|\frac{-x}{10}\right|<1$

Converge en $|x|<10$

$$\frac{1}{10}\sum_{n=0}^{+\infty}\left(-\frac{x}{10}\right)^n=\boxed{\frac{1}{10}\sum_{n=0}^{+\infty}(-1)^n\cdot\left(\frac{x}{10}\right)^n}$$

(g) $\frac{1}{4-x^2}=f(x)=\frac{1}{4}\cdot\frac{1}{1-\left(\frac{x^2}{4}\right)}=\frac{1}{4}\cdot\left(1+\frac{x^2}{4}+\frac{x^4}{16}+\frac{x^6}{64}+\dots+\left(\frac{x^2}{4}\right)^n\right)$

$$\boxed{\frac{1}{4}\cdot\sum_{n=0}^{+\infty}\left(\frac{x^2}{4}\right)^n}=$$ $\left|\frac{x^2}{4}\right|<1$, $|x|<\sqrt{4}=2$

(k) $\frac{x}{(1+x^2)^2}=f(x)=\frac{x}{(1+x^2)}\cdot\frac{1}{(1+x^2)}$

$$\frac{1}{1-(-x^2)}=\frac{1}{1+x^2}\Rightarrow\left(\frac{1}{1+x^2}\right)'=\frac{-1(2x)}{(1+x^2)^2}=-2\cdot\frac{x}{(1+x^2)^2}$$

$\lim_{n\to\infty}\left|\frac{n+1\cdot x\cdot(x^2)^{n+1}}{n+1\cdot x\cdot(x^2)^n}\right|$, $|x^2|$, $|x|^2<1$, $|x|<1$

$$\sum_{n=1}(-x^2)^{n-1}=\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1-x^2+x^4-x^6+\dots+(-x^2)^n$$

$$\left(\frac{1}{1+x^2}\right)'=0-2x+4x^3-6x^5+\dots+n(-x^2)^{n-1}\cdot-2x$$

$$-2\left(x-2x^3+3x^5+\dots+n.x(-x^2)^{n-1}\right)$$

$$-2\cdot\frac{x}{(1+x^2)^2}=-2\sum_{n=1}^{+\infty}n.x(-x^2)^{n-1}\Rightarrow\frac{x}{(1+x^2)^2}=\boxed{\sum_{n=0}^{+\infty}(n+1).x.(-1)^n.(x^2)^n}$$

$$\sum_{n=0}^{+\infty}(n+1).x.(-x^2)^n \quad (-1)^n(x^2)^n$$

(h) $f(x) = \frac{e^x - 1}{x}$

$f(x) = \frac{e^x}{x}$

$f'(x) = \frac{e^x . x - e^x . 1}{x^2} = \frac{e^x (x-1)}{x^2}$

$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\frac{e^x - 1}{x} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$

$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots + \frac{x^{n-1}}{n!}$

$\boxed{\sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!}}$

$\lim_{n\to\infty} \left| \frac{x^n \cdot n!}{(n+1)! \cdot x^{n+1}} \right| = 0$

$0 < 1$

$R = +\infty$ $\quad \forall x \in \mathbb{R}$

$\Rightarrow (-\infty, \infty)$ intervalo de convergência

(e) $\cos^2(x) = f(x) = \frac{1 + \cos(2x)}{2}$

$\cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos(2x) = \sum_{n=0}^{+\infty} \frac{(-1)^n . (2x)^{2n}}{(2n)!} = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \dots$

$1 + \cos(2x) = 1 + 1 - \frac{4x^2}{2!}$

$\frac{1 + \cos(2x)}{2} = \underbrace{\frac{1}{2} + \frac{1}{2}}_{1} - \frac{2x^2}{2!} + \frac{8x^4}{4!} - \frac{32x^6}{6!} + \dots + \frac{(-1)^n . 2^{2n-1} x^{2n}}{2n!}$

(f) $f(x) = (1+x) . e^{-x}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} . \frac{x^n}{n}$

$(1+x) = e^{(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots)}$

$\frac{(1+x)}{e^x} = \frac{e^{(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots)}}{e^x}$

$\frac{x^2}{2} - x$

$\frac{x^2 - 2x}{2}$

$(1+x) . e^{-x} = e^{(-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots)}$

$= \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{x^{n-1}}}$

(j) $f(x) = \arctan(x)$

$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n . (x^2)^n = 1 - x^2 + x^4 - x^6$

$\int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} \int (-1)^n (x^2)^n = \int 1 dx - \int x^2 dx + \int x^4 dx - \int x^6 dx$

$\arctan(x) = \frac{(x^2)^{n+1}}{n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

$\arctan(x) = \boxed{\sum_{n=1}^{+\infty} (-1)^{n+1} . \frac{x^{2n-1}}{2n-1}}$

5. (a) $\cos(10^\circ)$

$\cos\left(\frac{\pi}{18}\right)$

180° —— π

10° —— $\frac{\pi \cdot 10}{180} = \frac{\pi}{18}$

$\cos\left(\frac{\pi}{18}\right) = 0{,}9848077$

$$\cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$1 - \frac{(\pi/18)^2}{2!} + \frac{(\pi/18)^4}{4!} = 0{,}9848077$$

error $< 0{,}000000024$

(b) $\operatorname{sen}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

180° —— π

18° —— $\frac{\pi \cdot 18}{180} = \frac{\pi}{10}$

$$\pi/10 - \frac{(\pi/10)^3}{6} = 0{,}3089915$$

$\operatorname{sen}(\pi/10) = 0{,}3090169$

error $= \underbrace{0{,}0000}_{4\text{ cifras}}254$

(c) $\arctan(1/5) = x - \frac{x^3}{3} + \frac{x^5}{5}$

$\arctan\left(\frac{1}{5}\right) = 0{,}1973955$

$$\frac{1}{5} - \frac{0{,}008}{3} + \frac{0{,}00032}{5} = 0{,}1973973$$

error $= 0{,}0000017$

(d) $\operatorname{Ln}(5)$

$$5 = \frac{1+x}{1-x}$$

$5 - 5x = 1 + x$

$4 = 6x$

$\boxed{\frac{2}{3} = x}$

$$\operatorname{Ln}\left(\frac{1+x}{1-x}\right) = 2\left(2/3 + \frac{8/27}{3} + \frac{32/243}{5} + \frac{64/2187}{7} + \frac{128/19683}{9}\right.$$

$$\left(\frac{2}{3} + \frac{8}{81} + \frac{32}{1215} + \frac{64}{15309} + \dots + \frac{256}{177147 \cdot 11}\right.$$

$$2\left(\frac{2}{3} + \frac{8}{81} + \frac{32}{1215} + \frac{128}{15309} + \cdots\right)$$

$$\begin{array}{r} 1{,}6002613 \\ - \operatorname{Ln} 5 \\ \hline 0{,}0091 \end{array} < 10^{-3}$$

(e) $\sqrt{e} = e^{1/2}$

$$1 + 1/2 + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{6} + \frac{(1/2)^4}{24} + \frac{(1/2)^5}{120}$$

$$1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} \quad 1{,}6486979$$

error $< 10^{-4}$

6. (a) 1. $y' = x^2 + y^2 \qquad y(0) = 1$

$$\sum_{n=0}^{\infty} a_n x^n = y \Rightarrow y^2 = \sum_{n=0}^{\infty} a_n x^{2n}$$

$$\sum_{n=1}^{\infty} a_n n\, x^{n-1} = y'$$

$$\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} - x^2 - \sum_{n=0}^{\infty} a_n x^{2n} = \emptyset$$

$$\sum_{n=0}^{\infty} a_{n+1}(n+1) x^n - \sum_{n=0}^{\infty} a_n x^{n+n} = x^2$$