# PRACTICA 6: Teorías de Gauge y sus Diagramas de Feynman

1.

$$S = \int d^4x \left( D^*_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi \right)$$

con $D_\mu = \partial_\mu + ieA_\mu$

transformación $U(1)$ global:

$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)$

$\alpha$ constante

a)

$$D^*_\mu \phi'^* D^\mu \phi' - m^2 \phi'^* \phi' =$$

$$D^*_\mu e^{-i\alpha}\phi^* \; D^\mu e^{i\alpha}\phi - m^2 e^{-i\alpha}\phi^* e^{i\alpha}\phi =$$

es una constante y conmuta con $D^\mu$

$$D^*_\mu \underbrace{e^{-i\alpha}e^{i\alpha}}_{1} \phi^* D^\mu \phi - m^2 \underbrace{e^{-i\alpha}e^{i\alpha}}_{1} \phi^*\phi = D^*_\mu \phi^* D^\mu \phi - m^2 \phi^*\phi$$

$\Rightarrow$ $\boxed{\text{la acción } S \text{ es invariante}}$

b) Sea $\alpha = \alpha(x^\mu) \to$ no conmutará con $D_\mu$

$$D_\mu e^{i\alpha(x)} \neq e^{i\alpha(x)} D_\mu$$

$$D'^*_\mu \phi'^* D'^\mu \phi' - m^2 \phi'^* \phi' =$$

$$D'^*_\mu e^{-i\alpha}\phi^* D'^\mu e^{i\alpha}\phi - m^2 e^{-i\alpha}\phi^* e^{i\alpha}\phi =$$

$$(\partial_\mu - ieA'_\mu)e^{-i\alpha}\phi^*(\partial^\mu + ieA'^\mu)e^{i\alpha}\phi - m^2 e^{-i\alpha}\phi^* e^{i\alpha}\phi =$$

$$\left[\partial_\mu(e^{-i\alpha}\phi^*) - ieA'_\mu e^{-i\alpha}\phi^*\right]\left[\partial^\mu(e^{i\alpha}\phi) + ieA'^\mu e^{i\alpha}\phi\right]$$
$$- m^2 \underline{e^{-i\alpha}\phi^*}\, \underline{e^{i\alpha}\phi} = \text{ conmutan}$$

$$\left[-e^{-i\alpha} i\partial_\mu\alpha\,\phi^* + e^{-i\alpha}\partial_\mu\phi^* - ieA'_\mu e^{-i\alpha}\phi^*\right].$$
$$\left[e^{i\alpha} i\partial^\mu\alpha\,\phi + e^{i\alpha}\partial^\mu\phi + ieA'^\mu e^{i\alpha}\phi\right] - m^2\phi^*\phi =$$

$$e^{-i\alpha}\left[-i\partial_\mu\alpha\,\phi^* + \partial_\mu\phi^* - ieA'_\mu\phi^*\right].$$
$$e^{i\alpha}\left[i\partial^\mu\alpha\,\phi + \partial^\mu\phi + ieA'^\mu\phi\right] - m^2\phi^*\phi =$$

$$\left(\partial_\mu\phi^* - i\left[eA'_\mu + \partial_\mu\alpha\right]\phi^*\right)\left(\partial^\mu\phi + i\left[eA'^\mu + \partial^\mu\alpha\right]\phi\right)$$
$$- m^2\phi^*\phi =$$

Comparemos esta expresión con la del $\mathcal{L}$ inicial (sin transformar)

$$(\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi =$$

$\Rightarrow$ requeriré para la invarianza:

$$e A'_\mu + \partial_\mu \alpha = e A_\mu$$

$$e A'_\mu = e A_\mu - \partial_\mu \alpha$$

$$\boxed{A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha} \quad \text{con} \quad \boxed{\phi' = e^{i\alpha}\phi}$$

c) El Lagrangiano de Maxwell es:

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

$$\mathcal{L} = -\frac{1}{16\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$\mathcal{L}' = -\frac{1}{16\pi}\left(\partial^\mu[A^\nu - \partial^\nu\lambda] - \partial^\nu[A^\mu - \partial^\mu\lambda]\right)\left(\partial_\mu[A_\nu - \partial_\nu\lambda] - \partial_\nu[A_\mu - \partial_\mu\lambda]\right)$$

donde hemos absorbido el $1/e$ en $\lambda$; $\frac{\alpha}{e} = \lambda$ $\Rightarrow$

$$\mathcal{L}' = -\frac{1}{16\pi}\Big(\partial^\mu A^\nu \underbrace{- \partial^\mu\partial^\nu\lambda - \partial^\nu A^\mu + \partial^\nu\partial^\mu\lambda}_{\text{su suma es nula}}\Big)\Big(\partial_\mu A_\nu - \partial_\nu A_\mu \underbrace{- \partial_\mu\partial_\nu\lambda + \partial_\nu\partial_\mu\lambda}_{=0}\Big)$$

$$\mathcal{L}' = -\frac{1}{16\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$\mathcal{L}' = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \quad \Rightarrow$$

d)

$$\underbrace{D^*_\mu \phi^* D^\mu \phi} - m^2 \phi^*\phi + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{3!}\phi^6$$

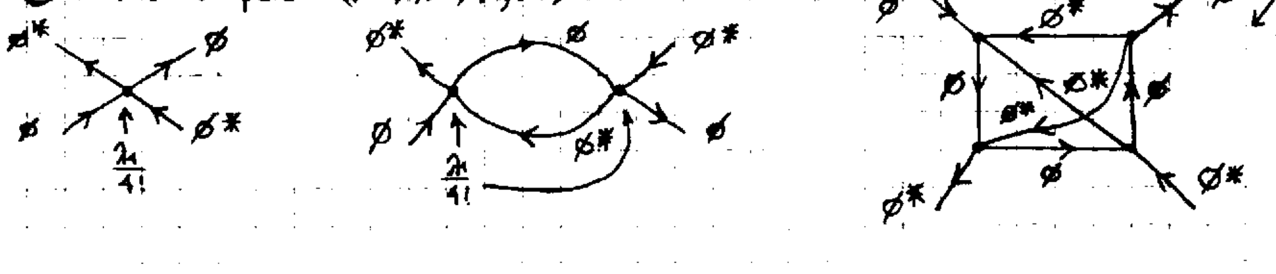
$$(\partial_\mu - iq A_\mu)\phi^*(\partial^\mu + iqA^\mu)\phi = \partial_\mu\phi^*\partial^\mu\phi + \underbrace{iq A_\mu \phi^* \partial^\mu \phi}_{①} \underbrace{- iq\partial_\mu\phi^* A^\mu\phi}_{②} + \underbrace{q^2 A_\mu\phi^* A^\mu \phi}_{③}$$

cinético para $\phi, \phi^*$

Proceso $\phi\phi^* \to \phi\phi^*$ (es un proceso de scattering)

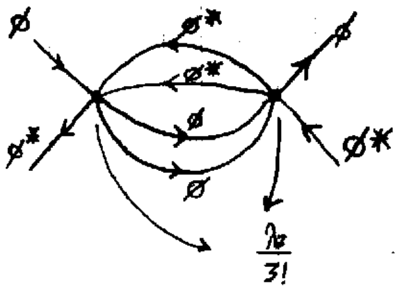
$$\mathcal{L} = \underbrace{D^*_\mu \phi^* D^\mu \phi}_{\text{t. cinético}} - \underbrace{m^2 \phi^*\phi}_{\text{t. de masa}} + \underbrace{\frac{\lambda_4}{4!}\phi^4}_{①} + \underbrace{\frac{\lambda_3}{3!}\phi^6}_{②}$$

① vértices de 4 patas $(\phi^*, \phi^*, \phi, \phi)$

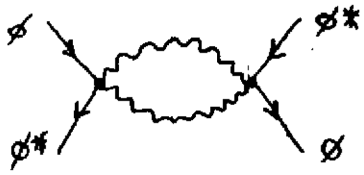



Tiene lazos en el diagrama. Cada vértice es de orden $\frac{\lambda_4}{4!}$

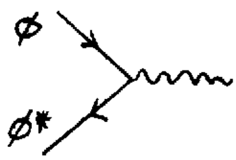
② vértices de 6 patas ($\phi^*, \phi^*, \phi^*, \phi, \phi, \phi$)




③ vértice de 4 patas ($\phi^*, A_\mu, A^\mu, \phi$)

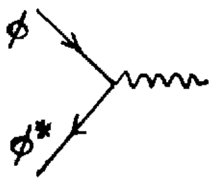



④ vértices de 3 patas ($A_\mu, \phi^*, \phi$)




Para el proceso en cuestión no los considero

⑤ vértices de 3 patas ($A_\mu, \phi^*, \phi$)




Para el proceso en cuestión no los considero

2.

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

con $D_\mu = \partial_\mu + ieA_\mu$ $\quad (-1/4)[\partial^\mu A^\nu - \partial^\nu A^\mu]. [\partial_\mu A_\nu - \partial_\nu A_\mu]$

$\lambda \equiv \frac{\alpha}{e}$ transf. conjunta $\rightarrow \begin{cases} \psi' = e^{i\alpha}\psi \\ A'_\mu = A_\mu - \partial_\mu \lambda \end{cases}$ $\qquad \psi^\dagger\gamma^0 = \bar{\Psi} \Rightarrow$

$$\psi'^\dagger = e^{-i\alpha}\psi^\dagger$$

$$D'_\mu = \partial_\mu + ieA'_\mu \qquad \bar{\Psi}' = e^{-i\alpha}\bar{\Psi}$$

$$D'_\mu = \partial_\mu + ieA_\mu - i\partial_\mu\alpha$$

a)

$$\mathcal{L}' = i\, e^{-i\alpha}\bar{\Psi}\gamma^\mu\left(\partial_\mu + ieA_\mu - i\partial_\mu\alpha\right)e^{i\alpha}\Psi - m\, e^{-i\alpha}\bar{\Psi}e^{i\alpha}\psi - \frac{1}{4}\left[\partial^\mu(A^\nu - \partial^\nu\lambda) - \partial^\nu(A^\mu - \partial^\mu\lambda)\right]\left[\partial_\mu(A_\nu - \partial_\nu\lambda) - \partial_\nu(A_\mu - \partial_\mu\lambda)\right]$$

$$= ie^{-i\alpha}\bar{\Psi}\gamma^\mu\left(e^{i\alpha}i\partial_\mu\alpha\Psi + e^{i\alpha}\partial_\mu\Psi + ieA_\mu e^{i\alpha}\Psi - i\partial_\mu\alpha\, e^{i\alpha}\Psi\right) - \frac{1}{4}\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right]\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right] - m\, e^{-i\alpha}\bar{\Psi}e^{i\alpha}\Psi$$

NOTA: La invariancia de $F^{\mu\nu}F_{\mu\nu}$ se probó en el problema 1

$$= i\bar{\Psi}\gamma^\mu \underbrace{e^{-i\alpha}e^{i\alpha}}_{=1}\left(\cancel{i\partial_\mu\alpha\Psi} + \partial_\mu\Psi + ieA_\mu\Psi - \cancel{i\partial_\mu\alpha\Psi}\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \; - m\underbrace{e^{-i\alpha}e^{i\alpha}}_{=1}\bar{\Psi}\Psi$$

$$= i\bar{\Psi}\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m\bar{\Psi}\Psi$$

$$\mathcal{L}' = i\bar{\Psi}\gamma^\mu D_\mu\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \mathcal{L} \Rightarrow \boxed{\text{ha resultado } \mathcal{L} \text{ invariante}}$$

b)

$$\Psi \rightarrow e^{i\alpha\gamma^5}\psi \qquad A_\mu \rightarrow A_\mu$$

$$\psi' = e^{i\alpha\gamma^5}\psi = \sum_{k=0}^{\infty}\frac{(i\alpha\gamma^5)^k}{k!}\psi \Rightarrow$$

$$(\psi')^\dagger = \sum_{k=0}^{\infty}\frac{\psi^\dagger(-i\alpha\gamma^{5\dagger})^k}{k!} = \psi^\dagger e^{-i\alpha\gamma^5}$$

$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \rightarrow \gamma^{5\dagger} = \gamma^5 \quad \mathbb{1}^\dagger = \mathbb{1}$ $\qquad \{\gamma^5, \gamma^0\} = 0 \rightarrow$

$(\gamma^5)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad (\gamma^5)^{2k} = \mathbb{1}$, $\quad (\gamma^5)^{2k+1} = \gamma^5$ $\qquad \gamma^5\gamma^0 = -\gamma^0\gamma^5 \rightarrow$

$$e^{-i\alpha\gamma^5}\gamma^0 = \sum_{k=0}^{\infty} \frac{(-i\alpha\gamma^5)^k}{k!}\gamma^0$$

$k$ par $(\gamma^5)^k = \mathbb{1} \rightarrow \gamma^5\gamma^0 = \gamma^0\gamma^5$

$k$ impar $(\gamma^5)^k = \gamma^5 \rightarrow \gamma^5\gamma^0 = -\gamma^0\gamma^5$

$k$ par $(-i\alpha\gamma^5) = i\alpha\mathbb{1} \rightarrow (-i\alpha\gamma^5)^k\gamma^0 = \gamma^0 (i\alpha\gamma^5)^k$

$k$ impar $(-i\alpha\gamma^5) = -i\alpha\gamma^5 \rightarrow (-i\alpha\gamma^5)^k\gamma^0 = \gamma^0(i\alpha\gamma^5)^k$

$$e^{-i\alpha\gamma^5}\gamma^0 = \sum_{k=0}^{\infty}\frac{(-i\alpha\gamma^5)^k}{k!}\gamma^0 = \gamma^0\sum_{k=0}^{\infty}\frac{(i\alpha\gamma^5)^k}{k!} = \gamma^0 e^{i\alpha\gamma^5}$$

$$(\Psi')^\dagger\gamma^0 = \overline{\Psi}' = \overline{\Psi} e^{i\alpha\gamma^5} \Rightarrow$$

$$\mathcal{L}' = i\overline{\Psi}e^{i\alpha\gamma^5}\gamma^\mu D_\mu e^{i\alpha\gamma^5}\Psi - m\overline{\Psi}e^{i\alpha\gamma^5}e^{i\alpha\gamma^5}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$i\overline{\Psi}e^{i\alpha\gamma^5}\gamma^\mu(\partial_\mu + ieA_\mu)e^{i\alpha\gamma^5}\Psi$$

$$\mathcal{L}' = i\overline{\Psi}e^{i\alpha\gamma^5}\gamma^\mu e^{i\alpha\gamma^5} D_\mu\Psi - m\overline{\Psi}\Psi e^{2i\alpha\gamma^5} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Como $\{\gamma^5, \gamma^\mu\} = 0 \rightarrow \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \Rightarrow$

$$e^{i\alpha\gamma^5}\gamma^\mu = \sum_{k=0}^{\infty}\frac{(i\alpha\gamma^5)^k}{k!}\gamma^\mu$$

Ⓐ con $k$ par $(i\alpha\gamma^5)^k\gamma^\mu = \gamma^\mu(i\alpha\gamma^5)^k$

Ⓑ con $k$ impar $(i\alpha\gamma^5)^k\gamma^\mu = -\gamma^\mu(i\alpha\gamma^5)^k$

podemos meter un $-1$ en Ⓐ porque se halla a potencia par $\Rightarrow$
podemos meter el $-1$ en Ⓑ dentro del paréntesis a la pot. impar $\Rightarrow$

$$e^{i\alpha\gamma^5}\gamma^\mu = \gamma^\mu\sum_{k=0}^{\infty}\frac{(-i\alpha\gamma^5)^k}{k!} = \gamma^\mu e^{-i\alpha\gamma^5} \Rightarrow$$

$$\mathcal{L}' = i\overline{\Psi}\gamma^\mu \underbrace{e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}}_{=1} D_\mu\Psi - m\overline{\Psi}\Psi e^{2i\alpha\gamma^5} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

si tomamos $m = 0 \Rightarrow$

$$\mathcal{L}' = i\overline{\Psi}\gamma^\mu D_\mu\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \mathcal{L}\ [\text{con } m=0]$$

$\boxed{\text{Es invariante sólo si el campo tiene masa nula}}$

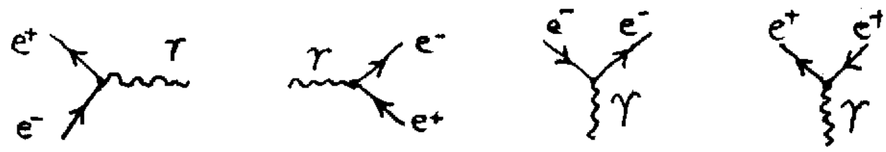
**3.**

*P2 a)

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \underbrace{D_\mu}_{\partial_\mu + ieA_\mu}\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

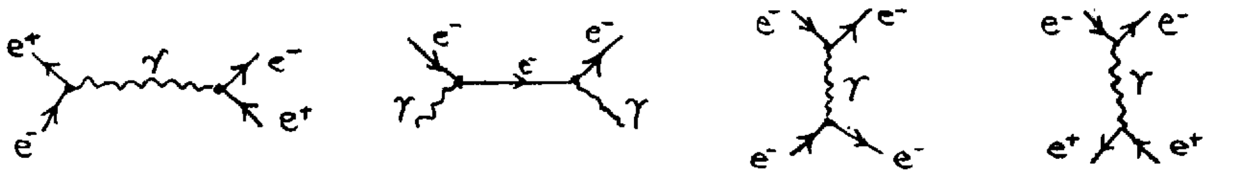
$$\mathcal{L} = \underbrace{i\bar{\Psi}\gamma^\mu\partial_\mu\Psi}_{\text{término cinético para } \Psi} \underbrace{+e\bar{\Psi}\gamma^\mu A_\mu\Psi}_{\text{t. de interacción}} \underbrace{-m\bar{\Psi}\Psi}_{\text{término cuadrático}} \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{término cinético para } A_\mu}$$

El único termino de interacción es: $e\bar{\Psi}\gamma^\mu A_\mu\Psi$ que implica vértices de 3 patas que aportarán $ie\gamma^\mu$ por cada vértice;




aniquilación creación scattering

Dependiendo del proceso en consideración podríamos tener:




$e^+e^- \longrightarrow e^+e^-$ $\gamma e^- \longrightarrow e^-\gamma$ scattering $e^-e^- \to e^-e^-$ scattering $e^-e^+ \to e^-e^+$


$e^+e^- \longrightarrow e^+e^-$ $e^+e^- \longrightarrow e^+e^-$ $e^+e^- \longrightarrow \gamma\gamma$

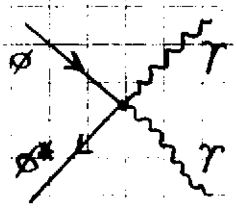
aniquilación


$e^+e^- \to \gamma\gamma$

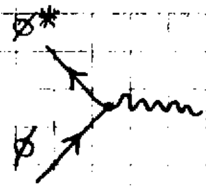
***P1.**

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - \underbrace{ie A_\mu \phi^*(\partial^\mu\phi)}_{③} + \underbrace{ie(\partial_\mu\phi^*)A^\mu\phi}_{②} + \underbrace{e^2 A_\mu\phi^* A^\mu\phi}_{\text{término de interacción } ①} - \underbrace{m^2\phi^*\phi}_{\text{término de masa (cuadrático)}}$$

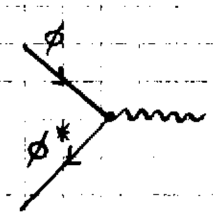
① Este término implica vértices de 4 patas que aportarán $ie^2$ por cada vértice a la amplitud




② Este término implica vértices de 3 patas que aportan $-e$ por cada vértice



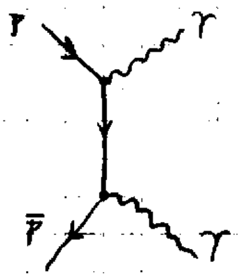

③ Este término representa vértices de 3 patas que aportan $e$ a la amplitud




b)

$p\bar{p} \longrightarrow \gamma\gamma$ con el $\mathcal{L}_{QED}$ (problema 2) será:

- vértices de 3 patas por el término de interacción
  $\Rightarrow ie\gamma^\mu$ por cada vértice




No hay diagramas con 1 vértice.
El 1er no nulo es el de 2 vértices

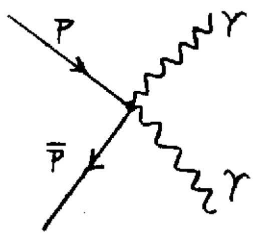
$\mathcal{M} \sim (ie\gamma^\mu)^2$ → aparece a orden $e^2$ el 1er término no nulo (será de dos vértices)

$p\bar{p} \to \gamma\gamma$ con el $\mathcal{L}_{ZS}$ (problema 1) será:

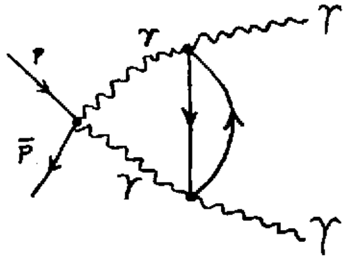
_vértices de 4 patas

$\Rightarrow$ $ie^2$ por cada vértice

$\mathcal{A} \sim ie^2$

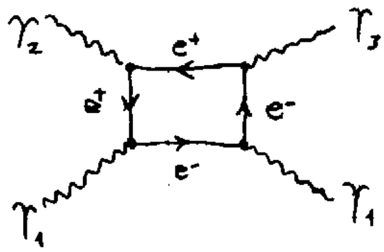



aparece a orden $e^2$ el 1er término no nulo (1 vértice)




$\mathcal{A} \sim (ie^2)^3 \sim e^6$ ← Esta es una corrección de orden mucho menor

c)




Jackson dice que el principio de incertidumbre permite la creación de un par $(e^+)(e^-)$ y luego desaparecen con la emisión de dos fotones diferentes.

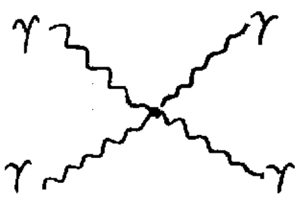
Las ondas planas de los fotones no se superponen linealmente sino que hay interacción y se transforman cambiando su momento

Como se ve, es indispensable la consideración de un lazo (el cuadrado recorrido cerradamente) para este proceso.

d) Un gráfico clásico para el proceso

$$\gamma + \gamma \to \gamma + \gamma$$

No debería incluir lazos $\Rightarrow$ podríamos pensarnos o soñarnos este:




(También se puede pensar como el caso límite en que el lazo se hace infinitesimal y se transforma en un punto)

el cual requeriría un término de la forma:

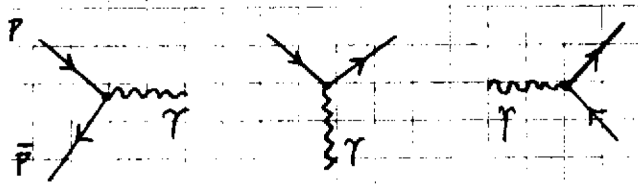
$$g(A_\mu A^\mu)^2 = g\, A_\mu A^\mu A_\mu A^\mu \quad , \; g \equiv \text{constante de acoplamiento}$$

que representa vértices elementales de 4 patas fotónicas. Para la constante $g$ que debería acompañarlo será: $g \sim e^4$, dado que el gráfico con lazo tiene cuatro vértices que aportan $e$ a la amplitud $\rightarrow$

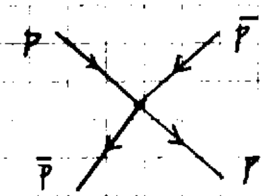
$$\mathcal{A} \sim e^4 \quad \rightarrow \quad \sigma \sim e^8$$

e) $$\mathcal{L} = \underbrace{i\bar{\psi}\gamma^\mu\partial_\mu\psi}_{} \underbrace{- e\bar{\psi}\gamma^\mu A_\mu\psi}_{①} - m\bar{\psi}\psi \underbrace{- g_1\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi}_{②} \underbrace{- g_2\bar{\psi}\gamma^\mu\psi\bar{\psi}\psi A_\mu}_{③} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

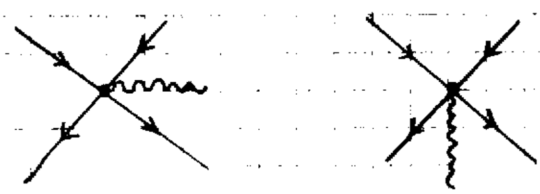
① típico término de interacción con vértices tripatas $(\bar{\psi}, A_\mu, \psi)$ que aporta $(ie\gamma^\mu)$ por vértice




② término de interacción de vértices con 4 patas $(\bar{\psi}, \psi, \bar{\psi}, \psi)$ que aporta $(ig_1\gamma^\mu\gamma_\mu)$ por vértice




③ término de interacción que da vértices de 5 patas $(\bar{\psi}, \psi, \bar{\psi}, \psi, A_\mu)$ que aporta $(ig_2\gamma^\mu)$ por vértice



4. No existe Problema 4

5.

$$\mathcal{L} = \text{Traza}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$A_\nu$ cuadrivector de matrices de $d \times d$

con $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$

$$A_\nu = A_\nu^a(x)\, J^a \quad \text{con} \quad [J^a, J^b] = i f^{abc} J^c$$

$$a = \{1, 2, 3, \ldots D\}$$

a) transformo el campo de gauge como:

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega^{-1}(x) - \frac{i}{g}\left(\partial_\mu \Omega(x)\right)\Omega^{-1}(x)$$

y transformo los campos de partículas según:

$$\psi \to \Omega(x)\psi \quad \text{con} \quad \Omega(x) = e^{i\omega_a(x) J^a}$$

⇒ quiero ver que el $\mathcal{L}_{YM}$ permanece invariante.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left(A_\mu A_\nu - A_\nu A_\mu\right)$$

conmutan (son # que dependen de x)

$$-ig\left(A_\mu^a A_\nu^b J^a J^b - A_\nu^b A_\mu^a J^b J^a\right) = -ig\, A_\mu^a A_\nu^b\, i f^{abc} J^c$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A_\mu^a A_\nu^b J^c$$
$$= \partial_\mu A_\nu^c J^c - \partial_\nu A_\mu^c J^c + g f^{cab} J^c A_\mu^a A_\nu^b$$
$$F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{cab} A_\mu^a A_\nu^b$$
$$F_c^{\mu\nu} = \partial^\mu A_c^\nu - \partial^\nu A_c^\mu + g f_{cab} A_a^\mu A_b^\nu$$
$$\Rightarrow \left(F_{\mu\nu}^c F_c^{\mu\nu}\right)' = {}_{①}\partial_\mu A_\nu'^c \partial^\mu A_c'^\nu - {}_{②}\partial_\mu A_\nu'^c \partial^\nu A_c'^\mu + g f_{cab} (\partial_\mu A_\nu'^c) A_a'^\mu A_b'^\nu$$
$$- \partial_\nu A_\mu'^c \partial^\mu A_c'^\nu + \partial_\nu A_\mu'^c \partial^\nu A_c'^\mu - g f_{cab} (\partial_\nu A_\mu'^c) A_a'^\mu A_b'^\nu$$
$$g f^{cab} A_\mu'^a A_\nu'^b \partial^\mu A_c'^\nu - g f^{cab} A_\mu'^a A_\nu'^b \partial^\nu A_c'^\mu + g^2 f^{cab} f_{cab} A_\mu^a A_\nu^b A_a^\mu A_b^\nu$$

$$A_\mu' = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} \qquad , \quad \partial_\mu \Omega = e^{i\omega_a J_a} \cdot i(\partial_\mu \omega_a) J_a$$
$$A_\mu' = \Omega A_\mu^a J^a \Omega^{-1} + \frac{1}{g}\Omega(\partial_\mu \omega_a) J^a \Omega^{-1} \qquad = \Omega\, i J_a (\partial_\mu \omega_a)$$
$$A_\mu'^c = \Omega A_\mu^c \Omega^{-1} + \frac{1}{g}\Omega(\partial_\mu \omega^c)\Omega^{-1} \longleftarrow$$

Veamos $(F_{\mu\nu}^c)'$

$$\partial_\mu A_\nu'^c - \partial_\nu A_\mu'^c = \underline{(\partial_\mu \Omega) A_\nu^c \Omega^{-1}} + \Omega(\partial_\mu A_\nu^c)\Omega^{-1} + \underline{\Omega A_\nu^c (\partial_\mu \Omega^{-1})} + \underline{\frac{1}{g}(\partial_\mu \Omega)(\partial_\nu \omega^c)\Omega^{-1}}$$
$$+ \frac{1}{g}\Omega(\cancel{\partial_\mu \partial_\nu} \omega^c)\Omega^{-1} + \underline{\frac{1}{g}\Omega(\partial_\nu \omega^c)\partial_\mu \Omega^{-1}}$$
$$\underline{- (\partial_\nu \Omega) A_\mu^c \Omega^{-1}} - \Omega(\partial_\nu A_\mu^c)\Omega^{-1} - \underline{\Omega A_\mu^c (\partial_\nu \Omega^{-1})} - \frac{1}{g}(\partial_\nu \Omega)(\partial_\mu \omega^c)\Omega^{-1}$$
$$- \frac{1}{g}\Omega(\cancel{\partial_\nu \partial_\mu} \omega^c)\Omega^{-1} - \frac{1}{g}\Omega(\partial_\mu \omega^c)\partial_\nu \Omega^{-1}$$

pero $\partial_\mu \Omega = i \Omega J^a (\partial_\mu \omega^a)$

$$\partial_\mu \Omega^{-1} = -e^{-i\omega_a J_a} i(\partial_\mu \omega_a) J_a = -i \Omega^{-1} J^a (\partial_\mu \omega^a)$$

$$i \Omega J^a \cancel{(\partial_\mu \omega^a)} A_\nu^c \Omega^{-1} + \Omega(\partial_\mu A_\nu^c)\Omega^{-1} - \Omega A_\nu^c\, i \cancel{J^a} \Omega^{-1} (\partial_\mu \omega^a)$$
$$- i \Omega J^a (\partial_\nu \cancel{\omega^a}) A_\mu^c \Omega^{-1} - \Omega(\partial_\nu A_\mu^c)\Omega^{-1} + \Omega A_\mu^c\, i \cancel{J^a} \Omega^{-1} (\partial_\nu \omega^a)$$
$$+ \frac{1}{g} i \Omega J^a \cancel{(\partial_\mu \omega^a)} (\partial_\nu \omega^c)\Omega^{-1} - \frac{1}{g}\Omega \cancel{(\partial_\nu \omega^c)}\, i \Omega^{-1} J^a (\partial_\mu \omega^a)$$
$$- \frac{1}{g} i \Omega J^a \cancel{(\partial_\nu \omega^a)} (\partial_\mu \omega^c)\Omega^{-1} + i\frac{1}{g}\Omega \cancel{(\partial_\mu \omega^c)}\Omega^{-1} J^a (\partial_\nu \omega^a)$$

$\Rightarrow$ Esta parte es invariante

$$+ g f^{cab} A_\mu'^a A_\nu'^b = g f^{cab}\left(\Omega A_\mu^a \Omega^{-1} + \frac{1}{g}\Omega(\partial_\mu \omega^a)\Omega^{-1}\right)\left(\Omega A_\nu^b \Omega^{-1} + \frac{1}{g}\Omega(\partial_\nu \omega^b)\Omega^{-1}\right)$$
$$= g f^{cab}\Big[\Omega A_\mu^a A_\nu^b \Omega^{-1} + \frac{1}{g}\Omega(\partial_\mu \omega^a) A_\nu^b \Omega^{-1} + \frac{1}{g}\Omega A_\mu^a (\partial_\nu \omega^b)\Omega^{-1}$$
$$+ \frac{1}{g^2}\Omega(\partial_\mu \omega^a)(\partial_\nu \omega^b)\Omega^{-1}\Big]$$

$$\text{Traza}\left(F_{\mu\nu}F^{\mu\nu}\right) = F^{a}_{\mu\nu}F^{\mu\nu}_{a} \qquad \text{con} \quad a=\{1,2,\dots D\}$$

$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$
$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g f_{abc} A^\mu_b A^\nu_c$$

Bastará ver que la transformación deja invariante a $F^a_{\mu\nu}$ $\Rightarrow$

$$\partial_\mu A'_\nu = e^{i\omega_a J^a} i(\partial_\mu \omega_a) J^a A_\nu(x) e^{-i\omega_a J^a}$$
$$+ e^{i\omega_a J^a}\left[\partial_\mu A_\nu(x) e^{-i\omega_a J^a} - A_\nu(x) \cdot e^{-i\omega_a J^a} i(\partial_\mu \omega_a) J^a\right]$$
$$- \frac{i}{g}\left[\partial_\mu\left(\partial_\nu e^{i\omega_a J^a}\right) e^{-i\omega_a J^a} - \left(\partial_\nu e^{i\omega_a J^a}\right) e^{-i\omega_a J^a} i(\partial_\mu \omega_a) J^a\right]$$

uso que $(\partial_\mu \omega_a)\Omega^{-1} = \Omega^{-1}(\partial_\mu \omega_a)$ escalares

$$= e^{i\omega_a J^a}(\partial_\mu A_\nu) e^{-i\omega_a J^a} - \frac{i}{g}\left[\partial_\mu\left(e^{i\omega_a J^a} i(\partial_\nu \omega_a) J^a\right) e^{-i\omega_a J^a}\right.$$
$$\left. - e^{i\omega_a J^a} i(\partial_\nu \omega_a) J^a \cdot e^{-i\omega_a J^a} i(\partial_\mu \omega_a) J^a\right]$$
$$= \partial_\mu A_\nu - \frac{i}{g}\left[\left(e^{i\omega_a J^a} i(\partial_\mu \omega_a) J^a i(\partial \omega_a) J^a + e^{i\omega_a J^a} i(\partial_\mu \partial_\nu \omega_a)\right.\right.$$
$$\left.\left(e^{-i\omega_a J^a}\right) - e^{i\omega_a J^a} i(\partial_\nu \omega_a) J^a e^{-i\omega_a J^a} i(\partial_\mu \omega_a) J^a\right]$$

uso $[J^a, J^a]=0 \longrightarrow$
$$\partial_\mu A'_\nu = \partial_\mu A_\nu - \frac{i}{g}\left(e^{i\omega_a J^a} i(\partial_\mu \partial_\nu \omega_a) J^a e^{-i\omega_a J^a}\right)$$

$$\partial_\nu A'_\mu = \partial_\nu A_\mu - \frac{i}{g}\left(e^{i\omega_a J^a} i(\partial_\nu \partial_\mu \omega_a) J^a e^{-i\omega_a J^a}\right)$$

$$-g f^{abc} A'^b_\mu A'^c_\nu = -g f^{abc}\left(\Omega A^b_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1}\right)\left(\quad\right)$$
$$\left(\Omega A^c_\nu \Omega^{-1} - \frac{i}{g}(\partial_\nu \Omega)\Omega^{-1}\right)$$

$$= -g f^{abc}\left(\Omega A^b_\mu \Omega^{-1}\Omega A^c_\nu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1}\Omega A^c_\nu \Omega^{-1}\right.$$
$$\left. - \Omega A^b_\mu \Omega^{-1}\frac{i}{g}(\partial_\nu \Omega)\Omega^{-1} - \frac{1}{g^2}(\partial_\mu \Omega)\Omega^{-1}(\partial_\nu \Omega)\Omega^{-1}\right)$$
$$= -g f^{abc}\left(\Omega A^b_\mu A^c_\nu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega) A^c_\nu \Omega^{-1} - \frac{i}{g}\Omega A^b_\mu \Omega^{-1}(\partial_\nu \Omega)\Omega^{-1}\right.$$
$$\left. - \frac{1}{g^2}(\partial_\mu \Omega)\Omega^{-1}(\partial_\nu \Omega)\Omega^{-1}\right)$$

$$\partial_\mu \Omega = \Omega\, i(\partial_\mu \omega_a) J^a \rightarrow (\partial_\mu \Omega)\Omega^{-1} = i(\partial_\mu \omega_a) J^a$$

c)

Un término de masa tendría la forma:

$$\Omega = e^{i\omega_a J^a}$$
$$\partial_\mu \Omega = \Omega\, i\, \partial_\mu \omega_a J^a$$
$$i\,\Omega J^a \partial_\mu \omega_a$$

$$\text{ter. masa} \equiv m\, A_\mu A^\mu \quad \Rightarrow$$

$$(\text{ter. masa})' = m\, A'_\mu A'^\mu$$

$$(t.m)' = m\left(\Omega A^a_\mu J^a \Omega^{-1} + \frac{1}{g}\Omega J^a \partial_\mu \omega_a \Omega^{-1}\right)\left(\Omega A^b_\nu J^b \Omega^{-1} + \frac{1}{g}\Omega J^b (\partial^\mu \omega^b)\Omega^{-1}\right)$$

$$= m\Big[\Omega A^a_\mu A^\mu_\nu J^a J_b \Omega^{-1} + \frac{1}{g}\Omega J^a(\partial_\mu \omega_a) A^\mu_b J_b \Omega^{-1} + \frac{1}{g}\Omega A^a_\mu J^a J_b (\partial^\mu \omega^b)\Omega^{-1} + \frac{1}{g^2}\Omega J^a(\partial_\mu \omega^a) J^b (\partial^\mu \omega^b)\Omega^{-1}\Big]$$

Como puede verse; esto no es invariante de gauge.

d)

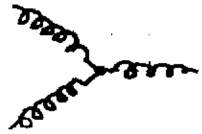
$$\mathcal{L} = \text{Traza}\left(F^{\mu\nu} F_{\mu\nu}\right) = \frac{1}{2} F^a_{\mu\nu} F^{\mu\nu}_a \longleftarrow$$

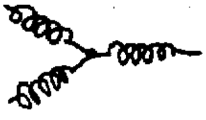
Esto es una notación: la traza en realidad sería $(F_{\mu\nu})_{ab}(F^{\mu\nu})_{ab}$. Cada elemento ab de la matriz es un tensor de 2do orden

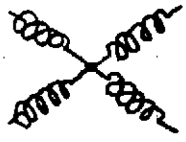
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$
$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^\mu_b A^\nu_c$$

$$\mathcal{L} = (\ldots) + \overbrace{g f^{abc} A^b_\mu A^c_\nu \partial^\mu A^\nu_a}^{1} - \overbrace{g f^{abc} A^b_\mu A^c_\nu \partial^\nu A^\mu_a}^{2} + \underbrace{g^2 f^{abc} f_{abc} A^b_\mu A^c_\nu A^\mu_b A^\nu_c}_{3} + \underbrace{g(\partial_\mu A^a_\nu) f_{abc} A^\mu_b A^\nu_c}_{4} - \underbrace{g(\partial_\nu A^a_\mu) f_{abc} A^\mu_b A^\nu_c}_{5}$$

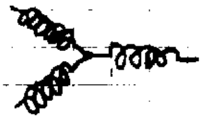
, donde $(\ldots)$ son términos cinéticos de $A^a_\nu$

①  con aporte de $\left(i g f^{abc}\right)$ en el vértice

②  " " " $\left(-i g f^{abc}\right)$ " " "

③  con aporte de $\left(i g^2 f^{abc} f_{abc}\right)$ en el vértice

④  con aporte de $\left(i g f_{abc}\right)$ en el vértice

⑤  con aporte de $(ig f_{abc})$ en el vértice

6.

a)
$$\mathcal{L} = \overbrace{i\bar{\Psi}^a\gamma^\mu (D_\mu)^{ab}\Psi^b - m\bar{\Psi}\Psi}^{\mathcal{L}\text{ de campos spin }1/2}\ \overbrace{-\frac{1}{4}\,\text{Traza}\left(F^{\mu\nu}F_{\mu\nu}\right)}^{\mathcal{L}_{YM}}$$

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + i\bar{\Psi}\gamma^\mu\, ig\, T_a A^a_\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}\text{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

donde
$$\Psi = \begin{pmatrix}\psi_R\\ \psi_G\\ \psi_B\end{pmatrix} = \begin{pmatrix} \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}_R \\ \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}_G \\ \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}_B \end{pmatrix}$$

b)
$$\underline{\underline{\Psi' = \Omega\Psi}} \qquad \text{con} \begin{cases}\Omega = \Omega(x)\\ \Psi = \Psi(x)\end{cases}$$

$$\Psi'^{\dagger} = (\Omega\Psi)^\dagger = \Psi^\dagger\Omega^\dagger \quad \rightarrow \quad \Psi'^{\dagger}\gamma^0 = \bar{\Psi}' = \Psi^\dagger\Omega^\dagger\gamma^0$$
$$= \Psi^\dagger\gamma^0\Omega^\dagger$$

pero $\Omega$ es unitaria $\rightarrow$

$$\Omega\Omega^\dagger = \mathbb{1} = \Omega\Omega^{-1} \Rightarrow \Omega^\dagger = \Omega^{-1} \quad \longrightarrow \quad \underline{\underline{\bar{\Psi}' = \bar{\Psi}\Omega^{-1}}}$$

$$(D'_\mu\Psi') = (\partial_\mu + igA'_\mu)(\Omega\Psi) = \partial_\mu(\Omega\Psi) + ig\,A'_\mu\Omega\Psi$$
$$= (\partial_\mu\Omega)\Psi + \Omega\,\partial_\mu\Psi + ig\,A'_\mu\Omega\Psi$$

$$\overline{\Psi}'\Psi' = \overline{\Psi}\,\Omega^{-1}\Omega\Psi = \overline{\Psi}\Psi \longrightarrow$$ el término de masa es invariante

$$\mathcal{L}' = \underbrace{i\overline{\Psi}\,\Omega^{-1}\gamma^\mu\partial_\mu(\Omega\Psi) - \Psi\Omega^{-1}\gamma^\mu g A'_\mu\Omega\Psi}\; - m\overline{\Psi}\Psi - \frac{1}{4}\mathrm{Tr}\left(\Omega F^{\mu\nu}F_{\mu\nu}\Omega^{-1}\right)$$

$$i\overline{\Psi}\,\Omega^{-1}\gamma^\mu(\partial_\mu\Omega)\Psi + i\overline{\Psi}\Omega^{-1}\underbrace{\gamma^\mu\Omega}_{\text{conmutan}}\partial_\mu\Psi - g\overline{\Psi}\,\Omega^{-1}\gamma^\mu A'_\mu\Omega\Psi$$

$$i\overline{\Psi}\,\Omega^{-1}\gamma^\mu(\partial_\mu\Omega)\Psi + \cancel{i\overline{\Psi}\gamma^\mu\partial_\mu\Psi} - g\,\overline{\Psi}\,\Omega^{-1}\gamma^\mu A'_\mu\Omega\Psi$$

$$= \cancel{i\overline{\Psi}\gamma^\mu\partial_\mu\Psi} - g\overline{\Psi}\gamma^\mu A_\mu\Psi$$

$$-g\overline{\Psi}\,\Omega^{-1}\gamma^\mu A'_\mu\Omega\Psi = -i\overline{\Psi}\Omega^{-1}\gamma^\mu(\partial_\mu\Omega)\Psi - g\overline{\Psi}\gamma^\mu A_\mu\Psi$$

$$\Omega^{-1}\gamma^\mu A'_\mu\Omega = \frac{i}{g}\Omega^{-1}\gamma^\mu(\partial_\mu\Omega) + \gamma^\mu A_\mu$$

$$\Omega^{-1}A'_\mu\Omega = A_\mu + \frac{i}{g}\Omega^{-1}(\partial_\mu\Omega)$$

Así transforma el gluón $\longrightarrow$ $$\boxed{A'_\mu = \Omega A_\mu\Omega^{-1} + \frac{i}{g}(\partial_\mu\Omega)\Omega^{-1}}$$

La traza es cíclica $\Rightarrow$

$$\mathrm{Tr}\left(F'^{\mu\nu}F'_{\mu\nu}\right) = \mathrm{tr}\left(\Omega F^{\mu\nu}F_{\mu\nu}\Omega^{-1}\right) = \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\Omega^{-1}\Omega\right) = \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

la traza es invariante de gauge

**7.**

$$-m\overline{\Psi}^a M^{ab}\Psi^b$$ $M$ es simétrica $\rightarrow M^{ab} = M^{ba}$

$$\Psi' = \Omega\Psi$$

$$\overline{\Psi}' = \overline{\Psi}\Omega^{-1} \rightarrow$$

$$-m\overline{\Psi}'M\Psi' = -m\overline{\Psi}\Omega M\Omega^{-1}\Psi$$ $\Longrightarrow$ será invariante el término de masa si $[\Omega, M] = 0$

$$\Omega = e^{i\omega_a T^a} \qquad M = \begin{pmatrix} b & f & g \\ f & c & h \\ g & h & d \end{pmatrix}$$

$\Omega$ es unitaria $\rightarrow$

$$\Omega^+\Omega = \mathbb{1}$$

$$\Omega^+ = \Omega^{-1}$$

$M$ es simétrica, y si los coeficientes son reales $\Rightarrow$ $M^+ = M \rightarrow$ es $M$ hermítica

Sea que $\Omega M = M\Omega \Rightarrow$ debe valer a orden infinitesimal $\rightarrow$

$$T^a M = M T^a \quad , \text{con } (T^a)^+ = T^a$$

$$(T^a M)^\dagger = M^\dagger T^{a\dagger} = M T^a$$

$(MT^a)^\dagger = MT^a \rightarrow MT^a$ es hermítica ; y esto no vale

en general : por ej.
$$\begin{pmatrix} b & f & g \\ f & c & h \\ g & h & d \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{T^1} = \begin{pmatrix} f & b & 0 \\ c & f & 0 \\ h & g & 0 \end{pmatrix} \rightarrow \text{No es hermítica}$$

$$\Rightarrow [M, T^a] \neq 0 \rightarrow [\Omega, M] \neq 0 \Rightarrow$$

La invarianza de gauge se rompe con $M^{ab}$ genérica

Supongamos ahora:
$$M^{ab} = m_a \delta^{ab} \rightarrow$$

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{no es hermítico (salvo que } a = b)$$

Vemos que tampoco funcionará con esta matriz $M^{ab}$ básicamente porque sigue sin conmutar con las $\lambda \rightarrow$ debido a que la interacción de color mezcla componentes la única matriz de masa que sirve es

$$M^{ab} = m \cdot \delta^{ab}$$