Práctica 5: Formulación Lagrangiana

1.

$$(1) \qquad \partial^\mu \partial_\mu \Theta(\vec{x},t) = 2\lambda\, e^{2\Theta(\vec{x},t)} \qquad , \quad \lambda \text{ parámetro constante}$$

$$\partial_\mu \partial^\mu \Theta = 2\lambda . e^{2\Theta}$$

$$\partial_\alpha\left(\frac{\partial L}{\partial(\partial_\alpha \Theta)}\right) - \frac{\partial L}{\partial \Theta} = 0 \qquad\qquad \partial_\mu(\partial^\mu \Theta) - 2\lambda e^{2\theta} = 0$$

$$f(\partial^\mu\theta) + f(\theta) = 0$$

$$\frac{\partial L}{\partial \theta} = 2\lambda . e^{2\theta}$$

$$\partial L = 2\lambda\, e^{2\theta}\, \partial\theta$$

$$L = \cancel{2}\lambda \frac{e^{2\theta}}{\cancel{2}} + \text{constante}(\partial_\alpha\theta)$$

$$\text{si} \quad L = \#\, \partial_\mu\theta\, \partial^\mu\theta \rightarrow$$

$$L = \#.\eta_{\mu\nu}\, \partial^\nu\theta\, \partial^\mu\theta$$

$$\rightarrow \frac{\partial L}{\partial(\partial_\alpha\theta)} = \#\eta_{\mu\nu}\, 2\, \partial^\nu\theta\, \eta^{\mu\alpha} = \#\, 2\, \partial_\mu\theta\, \eta^{\mu\alpha} = \#\, 2\partial^\alpha\theta$$

$$\partial_\alpha\left[\frac{\partial L}{\partial(\partial_\alpha\theta)}\right] = \#2\, \partial_\alpha\partial^\alpha\theta \rightarrow \qquad \# = 1/2 \Rightarrow$$

$$\boxed{L = \frac{1}{2}\, \partial_\mu\theta\, \partial^\mu\theta + \lambda . e^{2\theta}}$$

Lagrangiano cuyas ecuaciones de Euler-Lagrange desembocan en (1)

2.

$$L = -\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi^* - \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi^* + m^2 \phi^* \phi + V(\varphi, \phi)$$

a) Las ecuaciones de Euler-Lagrange son:

$$\partial_\mu \left[\frac{\partial L}{\partial[\partial_\mu \phi]} \right] - \frac{\partial L}{\partial \phi} = 0$$

- campo ϕ

$$\frac{\partial L}{\partial \phi} = m^2 \phi^* + \frac{\partial V}{\partial \phi}$$

$$\frac{\partial L}{\partial(\partial_\mu \phi)} = -\frac{1}{2}\partial^\mu \phi^*$$

- campo φ

$$\frac{\partial L}{\partial \varphi} = \frac{\partial V}{\partial \varphi}$$

$$\frac{\partial L}{\partial(\partial_\mu \varphi)} = -\frac{1}{2}\partial^\mu \varphi^*$$

- campo ϕ^*

$$\frac{\partial L}{\partial \phi^*} = m^2 \phi$$

$$\frac{\partial L}{\partial(\partial_\mu \phi^*)} = -\frac{1}{2}\partial^\mu \phi$$

- campo φ^*

$$\frac{\partial L}{\partial \varphi^*} = 0$$

$$\frac{\partial L}{\partial(\partial_\mu \varphi^*)} = -\frac{1}{2}\partial^\mu \varphi$$

Podemos escribir explícitamente la densidad lagrangiana como:

$$L = -\frac{1}{2}\left[\partial_0\phi\,\partial_0\phi^* - \partial_1\phi\,\partial_1\phi^* - \partial_2\phi\,\partial_2\phi^* - \partial_3\phi\,\partial_3\phi^*\right] -\frac{1}{2}\left[\partial_0\varphi\,\partial_0\varphi^* - \partial_1\varphi\,\partial_1\varphi^* - \partial_2\varphi\,\partial_2\varphi^* - \partial_3\varphi\,\partial_3\varphi^*\right] + m^2\phi^*\phi + V(\varphi,\phi)$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial L}{\partial(\partial_0 \phi^*)} = -\frac{1}{2}\partial_0\phi \\ &\Rightarrow \frac{\partial L}{\partial(\partial_i \phi^*)} = \frac{1}{2}\partial_i\phi \quad, i=1,2,3 \end{aligned}\right\} \frac{\partial L}{\partial \partial_\mu \phi^*} = -\frac{1}{2}\partial^\mu \phi$$

$$\boxed{\begin{aligned} &\phi) \quad \partial_\mu\left(-\frac{1}{2}\partial^\mu \phi^*\right) - m^2\phi^* - \frac{\partial V}{\partial \phi} = 0 \\ &\varphi) \quad \partial_\mu\left(-\frac{1}{2}\partial^\mu \varphi^*\right) - \frac{\partial V}{\partial \varphi} = 0 \\ &\phi^*) \quad \partial_\mu\left(-\frac{1}{2}\partial^\mu \phi\right) - m^2\phi = 0 \\ &\varphi^*) \quad \partial_\mu\left(-\frac{1}{2}\partial^\mu \varphi\right) = 0 \end{aligned}}$$

Se considera a ϕ, ϕ^* como campos independientes. Lo mismo para φ, φ^*, aunque debe notarse que las ecuaciones de Euler-Lagrange de un campo y su conjugado no nos brindan nueva información.

b) Dado un $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t)$ el momento canónicamente conjugado a q_j se define como:

$$p_j \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

Ahora, con una densidad lagrangiana L tendremos:

$$\boxed{p_\phi = \frac{\partial L}{\partial \phi} = m^2 \phi^* + \frac{\partial V}{\partial \phi} = -\frac{1}{2} \partial_\mu \partial^\mu \phi^*}$$

$$\boxed{p_\varphi = \frac{\partial L}{\partial \varphi} = \frac{\partial V}{\partial \varphi} = -\frac{1}{2} \partial_\mu \partial^\mu \varphi^*}$$

c) Tomamos el reemplazo $\begin{matrix} \partial_\mu \to \partial_\mu - ieA_\mu \\ \partial^\mu \to \partial^\mu - ieA^\mu \end{matrix} \Rightarrow$

$$L = -\frac{1}{2}\left(\partial_\mu - ieA_\mu\right)\phi\left(\partial^\mu - ieA^\mu\right)\phi^* - \frac{1}{2}\left(\partial_\mu - ieA_\mu\right)\varphi\left(\partial^\mu - ieA^\mu\right)\varphi^*$$

$$+ m^2 \phi^* \phi + V(\phi, \varphi)$$

$$L = -\frac{1}{2}\left[\partial_\mu \phi \partial^\mu \phi^* - ieA_\mu \phi \partial^\mu \phi^* - ie \partial_\mu \phi A^\mu \phi^* + i^2 e^2 A_\mu \phi A^\mu \phi^*\right]$$

$$-\frac{1}{2}\left[\partial_\mu \varphi \partial^\mu \varphi^* - ieA_\mu \varphi \partial^\mu \varphi^* - ie \partial_\mu \varphi A^\mu \varphi^* + i^2 e^2 A_\mu \varphi A^\mu \varphi^*\right]$$

$$+ m^2 \phi^* \phi + V(\phi, \varphi)$$

$A_\mu = (\Phi, \vec{A})$ es el cuadripotencial

$$\frac{\partial L}{\partial \phi} = \frac{ie}{2}\left(\partial^\mu \phi^*\right) A_\mu + \frac{e^2}{2} A_\mu \left(A^\mu \phi^*\right) + m^2 \phi^* + \frac{\partial V}{\partial \phi}$$

$$\frac{\partial L}{\partial(\partial_\nu \phi)} = -\frac{1}{2} \partial^\mu \phi^* \delta^\nu_\mu + \frac{ie}{2}\left(A^\mu \phi^*\right) \delta^\nu_\mu$$

$$\partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu \phi)}\right) - \frac{\partial L}{\partial \phi} = -\frac{1}{2} \partial_\nu \partial^\nu \phi^* + \frac{ie}{2} \partial_\nu A^\nu \phi^* - \frac{ie}{2} \partial^\mu \phi^* A_\mu - \frac{e^2}{2} A_\mu A^\mu \phi^*$$

$$- m^2 \phi^* - \frac{\partial V}{\partial \phi} = 0$$

$$\boxed{p_\phi = -\frac{1}{2} \partial_\nu \partial^\nu \phi^* + \frac{ie}{2} \partial_\nu A^\nu \phi^*}$$

Este es el reemplazo que nos lleva al EM

3.

$$S = -\frac{1}{4}\int d^4x \left(F_{\mu\nu} F^{\mu\nu} - \lambda A^\mu A_\mu \right)$$

$$S = \int d^4x \left[-\frac{1}{4}\left(F_{\mu\nu} F^{\mu\nu} - \lambda A^\mu A_\mu \right)\right] \quad \text{pero} \quad S = \int d^4x \, . L \Rightarrow$$

$$L = -\frac{1}{4}\left(F_{\mu\nu} F^{\mu\nu} - \lambda A^\mu A_\mu \right)$$

a)

$$F_{\mu\nu} F^{\mu\nu} = \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu \right)$$

$$F_{\mu\nu} F^{\mu\nu} - \lambda A^\mu A_\mu = \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\nu A_\mu \partial^\nu A^\mu - \lambda A^\mu A_\mu$$

- $$\frac{\partial L}{\partial(\partial_p A^q)} = \frac{1}{4}\left(\frac{\partial\left[\partial_\mu A_\nu - \partial_\nu A_\mu \right]}{\partial(\partial_p A^q)} . \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) + \left[\partial_\mu \partial_\nu - \partial_\nu A_\mu \right] \frac{\partial\left(\partial^\mu A^\nu - \partial^\nu A^\mu \right)}{\partial(\partial_p A^q)} \right)$$

$$\partial^\mu A^\nu = \eta^{\mu\alpha} \partial_\alpha A^\nu \qquad \partial_\mu A_\nu = \partial_\mu \eta_{\nu\alpha} A^\alpha = \eta_{\nu\alpha} \partial_\mu A^\alpha$$
$$\partial^\nu A^\mu = \eta^{\nu\alpha} \partial_\alpha A^\mu \qquad \partial_\nu A_\mu = \partial_\nu \eta_{\mu\alpha} A^\alpha = \eta_{\mu\alpha} \partial_\nu A^\alpha$$

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_p A^q)} = \delta^p_\mu \eta_{\nu q} \quad , \quad \frac{\partial(\partial^\mu A^\nu)}{\partial(\partial_p A^q)} = \eta^{\mu p} \delta^\nu_q \quad , \quad \frac{\partial(\partial^\nu A^\mu)}{\partial(\partial_p A^q)} = \eta^{\nu p} \delta^\mu_q$$

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_p A^q)} = \eta_{\nu\alpha} \frac{\partial(\partial_\mu A^\alpha)}{\partial(\partial_p A^q)} = \eta_{\nu\alpha} \delta^p_\mu \delta^\alpha_q = \eta_{\nu q} \delta^p_\mu$$

$$= -\frac{1}{4}\left[\left(\delta^p_\mu \eta_{\nu q} - \delta^p_\nu \eta_{\mu q} \right)\left(F^{\mu\nu} \right) + \left(F_{\mu\nu} \right)\left(\delta^\nu_q \eta^{\mu p} - \delta^\mu_q \eta^{\nu p} \right) \right]$$
$$= -\frac{1}{4}\left[\eta_{\nu q} F^{p\nu} - \eta_{\mu q} F^{\mu p} + F_{\mu q} \eta^{\mu p} - F_{q\nu} \eta^{\nu p} \right]$$
$$= -\frac{1}{4}\left[\eta_{\nu q}\left(\partial^p A^\nu - \partial^\nu A^p \right) - \eta_{\mu q}\left(\partial^\mu A^p - \partial^p A^\mu \right) + \left(\partial_\mu A_q - \partial_q A_\mu \right)\eta^{\mu p} - \left(\partial_q A_\nu - \partial_\nu A_q \right)\eta^{\nu p} \right]$$

$$\Rightarrow \partial^p A_q - \partial_q A^p - \partial_q A^p + \partial^p A_q + \partial^p A_q - \partial_q A^p - \partial_q A^p + \partial^p A_q$$

$$\frac{\partial L}{\partial(\partial_p \partial^q)} = -\frac{1}{4}4\left(\partial^p A_q - \partial_q A^p \right) = \partial_q A^p - \partial^p A_q \quad \Rightarrow$$

$$\partial_p\left[\frac{\partial L}{\partial(\partial_p A^q)} \right] = \partial_p \partial_q A^p - \partial_p \partial^p A_q = \partial_p \eta_{q\nu} \partial^\nu A^p - \partial_p \partial^p \eta_{q\nu} A^\nu$$
$$= \eta_{q\nu} \partial_p\left(F^{\nu p} \right) = \partial_p F_q{}^p$$

- $$\frac{\partial L}{\partial A^q} = +\frac{1}{4}\left(\lambda \frac{\partial(A^\mu A_\mu)}{\partial A^q} \right) = \frac{\lambda}{4}\eta_{\mu\alpha}\frac{\partial(A^\mu A^\alpha)}{\partial A^q} = \frac{\lambda}{4}\eta_{\mu\alpha}\left[\delta^\mu_q A^\alpha + A^\mu \delta^\alpha_q \right]$$

$$= \frac{\lambda}{4}\left[\eta_{\mu\alpha}\,\delta^{\mu}_{q}\,A^{\alpha} + \eta_{\mu\alpha}\,A^{\mu}\,\delta^{\alpha}_{q}\right] = \frac{\lambda}{4}\left[\eta_{q\alpha}A^{\alpha} + \eta_{\mu q}A^{\mu}\right]$$

$$= \frac{\lambda}{4}\left(A_q + A_q\right) = \frac{\lambda}{2}A_q$$

Entonces la ecuación de Euler-Lagrange queda:

$$\partial_\rho F_q{}^{\rho} = \frac{\lambda}{2}A_q$$

$$\boxed{\partial_\rho F^{\rho}{}_{q} = -\frac{\lambda}{2}A_q}$$

$$\eta_{q\nu}F^{\nu\rho} = F_q{}^{\rho}$$
$$-\eta_{q\nu}F^{\nu\rho} = -F_q{}^{\rho}$$
$$\eta_{q\nu}F^{\rho\nu} = F^{\rho}{}_{q} = \partial^\rho A_q - \partial_q A^\rho$$

b) Con $\lambda = 0$ es

$$\partial_\rho F^{\rho}{}_{q} = \partial_\rho\left(\partial^\rho A^\tau - \partial^\tau A^\rho\right)\eta_{q\tau} = 0$$

$$\eta_{q\tau}\,\partial_\rho\partial^\rho A^\tau - \eta_{q\tau}\,\partial_\rho\partial^\tau A^\rho = 0$$

$$\partial_\rho\partial^\rho A_q - \partial_\rho\partial_q A^\rho = 0$$

$$\partial_\rho\partial^\rho A_q - \partial_q(\partial_\rho A^\rho) = 0$$

pero: $\partial_\rho A^\rho = \partial^\rho A_\rho = \left(\frac{\partial}{\partial t}, \vec\nabla\right)\left(\phi, -\vec A\right) = \underbrace{\frac{\partial\phi}{\partial t} + \vec\nabla\cdot\vec A}_{} = 0$ ← Esto es justamente el gauge de Lorentz

$\Rightarrow$ usando el gauge de Lorentz $\partial^\rho A_\rho = 0$ resulta

$$\partial_\rho\partial^\rho A_q = \boxed{\partial^\rho\partial_\rho A^\nu = 0}$$

c) con $\lambda = 0$ es:

$$\partial_\rho\partial^\rho A_q - \partial_\rho\partial_q A^\rho = 0 \quad = \partial_\rho\partial^\rho A_q - \partial_q\partial_\rho A^\rho = 0$$

$$\Box^2 A_q = 0$$

$$\frac{\partial^2\vec A}{\partial t^2} - \nabla^2\vec A = 0 \quad \Rightarrow$$ Son las ecuaciones de Maxwell en ausencia de fuentes

d) con $\lambda \neq 0$ es:

$$\partial_\rho\partial^\rho A_q - \partial_q\underbrace{\partial^\rho A_\rho}_{=0} = -\frac{1}{2}\lambda A_q$$

$$\partial_\rho\partial^\rho$$

$$\Box^2 \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial\partial^\rho A_q + \lambda/2\, A_q = 0$$

$$\left(\Box^2 + \lambda/2\right)A_q = 0 \quad \rightarrow$$ Recuerda a la ecuación de Klein-Gordon si asociamos $\sqrt{\lambda/2}$ a la masa m

la ecuación de Klein Gordon es

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0$$

4.

a) $$S = -\frac{1}{4}\int d^4x\,(F_{\mu\nu}F^{\mu\nu} - \lambda A^\mu A_\mu) - \frac{1}{4}\int d^4x\, \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}F^{\rho\sigma}$$

$$S = \int d^4x\left(-\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} - \lambda A^\mu A_\mu + \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}F^{\rho\sigma} \right]\right)$$

densidad lagrangiana extra

$$L^E = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)$$

$$\frac{\partial L^E}{\partial(\partial_\alpha A^\beta)} = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial[\partial^\mu A^\nu - \partial^\nu A^\mu]}{\partial(\partial_\alpha A^\beta)}(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - (\partial^\mu A^\nu - \partial^\nu A^\mu)\frac{\partial[\partial^\rho A^\sigma - \partial^\sigma A^\rho]}{\partial(\partial_\alpha A^\beta)}\right)$$

$$\frac{\partial L^E}{\partial(\partial_\alpha A^\beta)} = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\left([\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta](\quad) - (\quad)[\eta^{\rho\alpha}\delta^\sigma_\beta - \eta^{\sigma\alpha}\delta^\rho_\beta]\right)$$

$$= -\frac{1}{4}\Big[(\epsilon_{\mu\beta\rho\sigma}\eta^{\mu\alpha} - \epsilon_{\beta\nu\rho\sigma}\eta^{\nu\alpha})(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - (\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$(\epsilon_{\mu\nu\rho\beta}\eta^{\rho\alpha} - \epsilon_{\mu\nu\beta\sigma}\eta^{\sigma\alpha})\Big]$$

$$= -\frac{1}{4}\Big[\epsilon_{\mu\beta\rho\sigma}\underset{Ⓐ}{\eta^{\mu\alpha}}\partial^\rho A^\sigma - \epsilon_{\beta\nu\rho\sigma}\underset{Ⓑ}{\eta^{\nu\alpha}}\partial^\rho A^\sigma - \epsilon_{\mu\beta\rho\sigma}\underset{Ⓒ}{\eta^{\mu\alpha}}\partial^\sigma A^\rho + \epsilon_{\beta\nu\rho\sigma}\underset{Ⓓ}{\eta^{\nu\alpha}}\partial^\sigma A^\rho$$
$$- \partial^\mu A^\nu\epsilon_{\mu\nu\rho\beta}\eta^{\rho\alpha} + \partial^\mu A^\nu\epsilon_{\mu\nu\beta\sigma}\eta^{\sigma\alpha} + \partial^\nu A^\mu\epsilon_{\mu\nu\rho\beta}\eta^{\rho\alpha} - \partial^\nu A^\mu\epsilon_{\mu\nu\beta\sigma}\eta^{\sigma\alpha}\Big]$$

se pueden reescribir como:

$$-\partial^\mu A^\nu\epsilon_{\rho\beta\mu\nu}\eta^{\rho\alpha} + \partial^\mu A^\nu\epsilon_{\beta\sigma\mu\nu}\eta^{\sigma\alpha} + \partial^\nu A^\mu\epsilon_{\rho\beta\mu\nu}\eta^{\rho\alpha} - \partial^\nu A^\mu\epsilon_{\beta\sigma\mu\nu}\eta^{\sigma\alpha}$$

se cancela con Ⓐ — se cancela con Ⓑ — se cancela con Ⓒ — se cancela con Ⓓ

$$\boxed{\frac{\partial L^E}{\partial(\partial_\alpha A^\beta)} = 0} \rightarrow \partial_\alpha\left(\frac{\partial L^E}{\partial(\partial_\alpha A^\beta)}\right) = \frac{\partial L^E}{\partial A^\beta} = 0$$

El término agregado a la acción no aporta nada a las ecuaciones de movimiento

• NOTA:
Acá la clave es que en términos como:

$$\partial^\mu A^\nu \epsilon_{\rho\beta\mu\nu}\eta^{\rho\alpha}$$

en realidad los índices μ,ν,ρ están contraídos y son, por ende mudos, dicho término depende de los índices $\beta\alpha$ que al no estar contraídos no pueden reemplazarse por otros (NO son mudos).

Además en la notación tensorial son productitos de números, entonces:

$$\partial^\mu A^\nu \epsilon_{\rho\beta\mu\nu}\eta^{\rho\alpha} = \epsilon_{\rho\beta\mu\nu}\eta^{\rho\alpha}\partial^\mu A^\nu$$

b)

$$S = -\frac{1}{4}\int d^4x \left(\epsilon_{\mu\nu\rho\sigma}\, F^{\mu\nu} F^{\rho\sigma} \right)$$

$$F^{\mu\nu}F^{\rho\sigma} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \end{array} \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \end{array}$$

Es conveniente expresar primeramente la matriz $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}$

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}$$

$$\underset{\underline{23}\ \underline{32}}{\epsilon_{01\rho\sigma}} F^{01} + \underset{\underline{13}\ 31}{\epsilon_{02\rho\sigma}} F^{02} + \underset{\underline{12}\ \underline{21}}{\epsilon_{03\rho\sigma}} F^{03} + \underset{\underline{23}\ \underline{32}}{\epsilon_{10\rho\sigma}} F^{10} + \underset{\underline{13}\ 31}{\epsilon_{20\rho\sigma}} F^{20} + \underset{\underline{12}\ \underline{21}}{\epsilon_{30\rho\sigma}} F^{30}$$

$$\underset{\underline{03}\ 30}{\epsilon_{12\rho\sigma}} F^{12} + \underset{\underline{02}\ 20}{\epsilon_{13\rho\sigma}} F^{13} + \underset{\underline{03}\ 30}{\epsilon_{21\rho\sigma}} F^{21} + \underset{\underline{02}\ 20}{\epsilon_{31\rho\sigma}} F^{31} + \underset{01\ 10}{\epsilon_{23\rho\sigma}} F^{23} + \underset{\underline{01}\ 10}{\epsilon_{32\rho\sigma}} F^{32}$$

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu} = \begin{pmatrix} 0 & \underset{(+1)}{\epsilon_{2301}}F^{23} + \underset{(-1)}{\epsilon_{3201}}F^{32} & \underset{(+1)}{\epsilon_{3102}}F^{31} + \underset{(-1)}{\epsilon_{1302}}F^{13} & \underset{(+1)}{\epsilon_{1203}}F^{12} + \underset{(-1)}{\epsilon_{2103}}F^{21} \\ \underset{(-1)}{\epsilon_{2310}}F^{23} + \underset{(+1)}{\epsilon_{3210}}F^{32} & 0 & \underset{(+1)}{\epsilon_{0312}}F^{03} + \underset{(-1)}{\epsilon_{3012}}F^{30} & \underset{(-1)}{\epsilon_{0213}}F^{02} + \underset{(+1)}{\epsilon_{2013}}F^{20} \\ \underset{(-1)}{\epsilon_{3120}}F^{31} + \underset{(+1)}{\epsilon_{1320}}F^{13} & \underset{(-1)}{\epsilon_{0321}}F^{03} + \underset{(+1)}{\epsilon_{3021}}F^{30} & 0 & \underset{(+1)}{\epsilon_{0123}}F^{01} + \underset{(-1)}{\epsilon_{1023}}F^{10} \\ \underset{(-1)}{\epsilon_{1230}}F^{12} + \underset{(+1)}{\epsilon_{2130}}F^{21} & \underset{(+1)}{\epsilon_{0231}}F^{02} + \underset{(-1)}{\epsilon_{2031}}F^{20} & \underset{(-1)}{\epsilon_{0132}}F^{01} + \underset{(+1)}{\epsilon_{1032}}F^{10} & 0 \end{pmatrix}$$

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu} = \begin{pmatrix} 0 & -B_x - B_x & -B_y - B_y & -B_z - B_z \\ B_x + B_x & 0 & -E_z - E_z & E_y + E_y \\ +B_y + B_y & E_z + E_z & 0 & -E_x - E_x \\ +B_z + B_z & -E_y - E_y & E_x + E_x & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}$$

$$\Rightarrow \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 2\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} -\vec{B}\cdot\vec{E} & 0 & 0 & 0 \\ 0 & -\vec{B}\cdot\vec{E} & 0 & 0 \\ 0 & 0 & -\vec{B}\cdot\vec{E} & 0 \\ 0 & 0 & 0 & -\vec{B}\cdot\vec{E} \end{pmatrix} \Rightarrow$$

volviendo a la acción tenemos:

$$S = \int d^4x \left[\frac{1}{2}(\vec{B}\cdot\vec{E}) \right] \mathbb{1}$$

$$F_{\mu\nu}F^{\mu\nu} = \eta_{\nu\tau}F_{\mu}{}^{\tau}F^{\mu\nu} = \eta_{\nu\tau}\eta_{\mu\sigma}F^{\sigma\tau}F^{\mu\nu}$$

Nótese que $F_{\mu\nu}F^{\mu\nu}$ es la traza del producto de dos matrices $F_{\mu\nu}$

$$(\overleftrightarrow{A}\cdot\overleftrightarrow{B})_{ij} = A_{ik}B_{kj}$$

$$\text{Traza } (\overleftrightarrow{A}\cdot\overleftrightarrow{B})_{ij} = A_{ik}B_{ki} \Rightarrow$$ No es la traza, pero casi : con lo cual

$$F_{\mu\nu}F^{\mu\nu} = F_{00}F^{00} + F_{11}F^{11} + F_{22}F^{22} + F_{33}F^{33} + F_{10}F^{10} + F_{12}F^{12} + F_{13}F^{13}$$
$$+ F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + \ldots$$

$$F_{\mu\nu}F^{\mu\nu} = \underline{E_x^2} + \underline{E_y^2} + \underline{E_z^2} + \underline{E_x^2} + \underline{B_z^2} + \underline{B_y^2} + \underline{E_y^2} + \underline{B_z^2} + \underline{B_x^2} + \underline{E_z^2} + \underline{B_y^2} + \underline{B_x^2}$$
$$= 2(E_x^2 + E_y^2 + E_z^2) + 2(B_x^2 + B_y^2 + B_z^2) = 2(\vec{E}^2 + \vec{B}^2)$$

$$S = -\frac{1}{4}\int d^4x\, 2(\vec{E}^2 + \vec{B}^2) = \int d^4x \left[-\frac{1}{2}(\vec{E}^2 + \vec{B}^2)\right]$$

La energía electromagnética está dada por este último término

$$\mathcal{L} = T - V$$

$$\mathcal{L}_P = \frac{1}{2}(\vec{E}\cdot\vec{B}) - \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

¿

este es el aporte que proviene del potencial V en electromagnetismo

5.

$$L = i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

con $D_\mu = \partial_\mu - ieA_\mu$

Reescribimos L como:

$$L = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi \underbrace{-i^2}_{+1} e\bar{\Psi}\gamma^\mu A_\mu\Psi - m\bar{\Psi}\Psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

, los campos intervinientes serán $\Psi, \bar{\Psi}, A_\mu$

• Campo Ψ

$$\partial_\alpha\left(\frac{\partial L}{\partial(\partial_\alpha\Psi)}\right) = \partial_\alpha\left(i\bar{\Psi}\gamma^\mu\delta^\alpha_\mu\right) \qquad \frac{\partial L}{\partial\Psi} = e\bar{\Psi}\gamma^\mu A_\mu - m\bar{\Psi}$$

$$(1) \quad \boxed{i\partial_\mu\bar{\Psi}\gamma^\mu - e\bar{\Psi}\gamma^\mu A_\mu + m\bar{\Psi} = 0}$$

• Campo $\bar{\Psi}$

$$\partial_\alpha\left(\frac{\partial L}{\partial(\partial_\alpha\bar{\Psi})}\right) = 0 \qquad \frac{\partial L}{\partial\bar{\Psi}} = i\gamma^\mu\partial_\mu\Psi + e\gamma^\mu A_\mu\Psi - m\Psi$$

$$(2) \quad \boxed{-i\gamma^\mu\partial_\mu\Psi - e\gamma^\mu A_\mu\Psi + m\Psi = 0}$$

• Campo A^μ

$$\begin{aligned}(\partial^\mu A^\nu - \partial^\nu A^\mu) &= \eta^{\mu\tau}\partial_\tau A^\nu - \partial^\nu\eta^{\mu\tau}A_\tau \\ &= \eta^{\mu\tau}\eta^{\nu\sigma}\partial_\tau A_\sigma - \eta^{\nu\sigma}\partial_\sigma A_\tau \\ &= \eta^{\mu\tau}\eta^{\nu\sigma}[\partial_\tau A_\sigma - \partial_\sigma A_\tau]\end{aligned}$$

$$\begin{aligned}\frac{\partial(1/4\, F_{\mu\nu}F^{\mu\nu})}{\partial(\partial_p A^q)} &= -\frac{1}{4}\eta^{\mu\tau}\eta^{\nu\sigma}\, 2\,(\partial_\tau A_\sigma - \partial_\sigma A_\tau)\left[\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_p A^q)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_p A^q)}\right] \\ &= -\frac{1}{2}\eta^{\mu\tau}\eta^{\nu\sigma}(\partial_\tau A_\sigma - \partial_\sigma A_\tau)(\delta^p_\mu\eta_{\nu q} - \delta^p_\nu\eta_{\mu q}) \\ &= -\frac{1}{2}(\partial_\tau A_\sigma - \partial_\sigma A_\tau)(\eta^{p\tau}\eta^{\nu\sigma}\eta_{\nu q} - \eta^{\mu\tau}\eta^{p\sigma}\eta_{\mu q}) \\ &= -\frac{1}{2}\left[\partial^p A^\nu\eta_{\nu q} - \partial^\mu A^p\eta_{\mu q} - \partial^\nu A^p\eta_{\nu q} + \partial^p A^\mu\eta_{\mu q}\right] \\ &= -\frac{1}{2}\left[\partial^p A_q - \partial_q A^p - \partial_q A^p + \partial^p A_q\right] = \partial_q A^p - \partial^p A_q\end{aligned}$$

Acá usamos otro método para derivar y llegamos al mismo resultado

$$\frac{\partial L}{\partial A^q} = e\bar{\Psi}\gamma^\mu\delta^q_\mu\Psi = e\bar{\Psi}\gamma_q\Psi$$

$$\partial_\rho\left(\partial_q A^\rho - \partial^\rho A_q\right) - e\bar{\Psi}\gamma_q\Psi = 0$$

Pero sabemos que

$$F^{q\rho} = \partial^q A^\rho - \partial^\rho A^q$$

$$\eta_{q\alpha}F^{\alpha\rho} = \eta_{q\alpha}\partial^\alpha A^\rho - \eta_{q\alpha}\partial^\rho A^\alpha = \partial_q A^\rho - \partial^\rho A_q = F_q{}^\rho \Rightarrow$$

$$\partial_\rho F_q{}^\rho - e\bar{\Psi}\gamma_q\Psi = 0 \quad \text{, Multiplico por } -\eta^{\nu q}$$

$$-\eta^{\nu q}\partial_\rho F_q{}^\rho + \eta^{\nu q} e\bar{\Psi}\gamma_q\Psi = 0$$

$$(3) \quad \boxed{\partial_\rho F^{\rho\nu} + e\bar{\Psi}\gamma^\nu\Psi = 0}$$

- NOTAS

La ecuación (1) puede obtenerse dagueando la (2) y multiplicando a izquierda por γ^0

$$-i\gamma^\mu\partial_\mu\Psi - e\gamma^\mu A_\mu\Psi + m\Psi = 0$$

$$i\gamma^\mu\partial_\mu\Psi^\dagger\gamma^0 - e\gamma^\mu A_\mu\Psi^\dagger\gamma^0 + m\Psi^\dagger\gamma^0 = 0$$

$$i\gamma^\mu\partial_\mu\bar{\Psi} - e\gamma^\mu A_\mu\bar{\Psi} + m\bar{\Psi} = 0$$

$$i\partial_\mu\bar{\Psi}\gamma^\mu - e\bar{\Psi}\gamma^\mu A_\mu + m\bar{\Psi} = 0$$

Asimismo, desde la (2) se tiene:

$$-i\gamma^\mu\left(\partial_\mu\Psi + \frac{e}{i}A_\mu\Psi\right) + m\Psi = 0$$

$$i\gamma^\mu[\partial_\mu - ieA_\mu]\Psi - m\Psi = 0$$

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \quad \leftarrow \text{Ecuación de Dirac Covariante}$$

Por otro lado, (3) es el EM pues

$$e\bar{\Psi}\gamma^\nu\Psi = J^\nu \rightarrow$$

$$\partial_\rho F^{\rho\nu} + J^\nu = 0 \rightarrow \text{es el EM covariante con fuentes (dadas por } J^\nu\text{)}$$

6.

$$S = \int d^3x \ \epsilon_{\mu\nu\rho} A^\mu F^{\nu\rho}$$

$$F^{\nu\rho} = \partial^\nu A^\rho - \partial^\rho A^\nu$$

$$L = \epsilon_{\mu\nu\rho}\left(A^\mu \partial^\nu A^\rho - A^\mu \partial^\rho A^\nu\right)$$

$$\frac{\partial L}{\partial(\partial_\alpha A^\beta)} = \epsilon_{\mu\nu\rho} A^\mu \frac{\partial(\partial^\nu A^\rho)}{\partial(\partial_\alpha A^\beta)} - \epsilon_{\mu\nu\rho} A^\mu \frac{\partial(\partial^\rho A^\nu)}{\partial(\partial_\alpha A^\beta)}$$

$$= \epsilon_{\mu\nu\rho} A^\mu (\eta^{\nu\alpha})\delta^\rho_\beta - \epsilon_{\mu\nu\rho} A^\mu (\eta^{\rho\alpha})\delta^\nu_\beta$$

$$= \epsilon_{\mu\nu\beta} A^\mu \eta^{\nu\alpha} - \epsilon_{\mu\beta\rho} A^\mu \eta^{\rho\alpha}$$

$$= \epsilon_{\mu\nu\beta} A^\mu \eta^{\nu\alpha} + \epsilon_{\mu\rho\beta} A^\mu \eta^{\rho\alpha} = 2\,\epsilon_{\mu\tau\beta} A^\mu \eta^{\tau\alpha}$$

$$\frac{\partial L}{\partial A^\beta} = \epsilon_{\mu\nu\rho}\partial^\nu A^\rho \delta^\mu_\beta - \epsilon_{\mu\nu\rho}\partial^\rho A^\nu \delta^\mu_\beta$$

$$= \epsilon_{\beta\nu\rho}\partial^\nu A^\rho - \epsilon_{\beta\nu\rho}\partial^\rho A^\nu = \epsilon_{\beta\nu\rho} F^{\nu\rho} =$$

$$\partial_\alpha\left(\frac{\partial L}{\partial(\partial_\alpha A^\beta)}\right) - \frac{\partial L}{\partial A^\beta} = 2\,\epsilon_{\mu\tau\beta}\partial_\alpha A^\mu \eta^{\tau\alpha} - \epsilon_{\beta\nu\rho} F^{\nu\rho} = 0$$

$$2\epsilon_{\mu\tau\beta}\partial^\tau A^\mu - \epsilon_{\beta\nu\mu}\partial^\nu A^\mu + \epsilon_{\beta\nu\mu}\partial^\mu A^\nu$$

$$-2\epsilon_{\mu\tau\beta}\partial^\tau A^\mu - \epsilon_{\nu\mu\beta}\partial^\nu A^\mu + \epsilon_{\nu\mu\beta}\partial^\mu A^\nu$$

$$-\epsilon_{\mu\tau\beta}\partial^\tau A^\mu - \epsilon_{\nu\mu\beta}\partial^\nu A^\mu$$

$$\epsilon_{\tau\mu\beta}\partial^\tau A^\mu - \epsilon_{\nu\mu\beta}\partial^\nu A^\mu = 0$$

$$\boxed{\epsilon_{\tau\mu\beta}\left(\partial^\tau A^\mu - \partial^\tau A^\mu\right) = 0}$$

• <u>Notar que</u>: $\epsilon_{\beta\nu\rho} F^{\nu\rho} = \epsilon_{\beta\nu\rho}\left(\partial^\nu A^\rho - \partial^\rho A^\nu\right) = \epsilon_{\beta\nu\rho}\partial^\nu A^\rho + \epsilon_{\beta\rho\nu}\partial^\rho A^\nu$

$$= 2\,\epsilon_{\beta\nu\rho}\partial^\nu A^\rho$$

$$\epsilon_{\tau\mu\beta}\left(F^{\tau\mu}\right) = 0$$

$$\boxed{F^{\tau\mu} = 0}$$