Práctica 4: Ecuaciones de onda relativistas, ecuación de Dirac

1.

$$\vec{p} = -i\hbar\vec{\nabla} \quad , \quad E = i\hbar\frac{d}{dt} \qquad \text{en} \qquad E = \frac{\vec{p}\cdot\vec{p}}{2m} \leftarrow$$ (energía de una partícula libre en física clásica NR)

$$i\hbar\frac{\partial}{\partial t} \doteq \frac{1}{2m}\hbar\,\underbrace{\vec{\nabla}\cdot\vec{\nabla}}_{\text{el escalar entre grad. da el laplaciano}} = \frac{\hbar^2\nabla^2}{2m}$$

Para partícula libre, en el caso relativista tenemos

$$\frac{E^2}{c^2} = p^2 + m^2c^2$$

$$\rightarrow \quad \frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)\left(i\hbar\frac{\partial}{\partial t}\right) \doteq \left(-i\hbar\vec{\nabla}\right)\left(-i\hbar\vec{\nabla}\right) + m^2c^2$$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} \doteq -\hbar^2\nabla^2 + m^2c^2$$

$$\boxed{\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\nabla^2 - m^2c^2\right)\psi = 0}$$

Relación de dispersión

si usamos $\hbar = c = 1 \rightarrow$

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\psi = \left(-\Box^2 - m^2\right)\psi = 0$$

$$\boxed{\left(\Box^2 + m^2\right)\psi = 0}$$

Ecuación de KG

2. Para la partícula libre tenemos; usando Schrödinger, que:

$$[1] \quad i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m}\psi \quad \Rightarrow \quad -i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\psi^* \quad [2]$$

$\psi^*.[1] - \psi.[2] =$

$$\psi^* i\hbar\frac{\partial\psi}{\partial t} = -\psi^*\frac{\hbar^2\nabla^2\psi}{2m}$$

$$-\quad -\psi\, i\hbar\frac{\partial\psi^*}{\partial t} = -\psi\frac{\hbar^2\nabla^2\psi^*}{2m}$$

$$i\hbar\left[\psi^*\partial_t\psi + \psi\partial_t\psi^*\right] = \frac{\hbar^2}{2m}\left[-\psi^*\nabla^2\psi + \psi\nabla^2\psi^*\right]$$

$$i\hbar\partial_t\left[\psi^*.\psi\right] = \frac{\hbar^2}{2m}\left[\psi\vec{\nabla}.(\vec{\nabla}\psi^*) - \psi^*\vec{\nabla}.(\vec{\nabla}\psi)\right]$$

usando

$$\vec{\nabla}.(f\vec{F}) = f\vec{\nabla}.\vec{F} + \vec{F}.\vec{\nabla}f$$ (identidad vectorial en libros)

$$\vec{\nabla}.(\psi^*\vec{\nabla}\psi) = \psi^*\vec{\nabla}.\vec{\nabla}\psi + \vec{\nabla}\psi.\vec{\nabla}\psi^* = \psi^*\nabla^2\psi + \vec{\nabla}\psi.\vec{\nabla}\psi^*$$

$$\vec{\nabla}.(\psi^*\vec{\nabla}\psi) - \vec{\nabla}\psi.\vec{\nabla}\psi^* = \psi^*\nabla^2\psi \quad \Rightarrow$$

$$\vec{\nabla}.(\psi\vec{\nabla}\psi^*) - \vec{\nabla}\psi^*.\vec{\nabla}\psi = \psi\nabla^2\psi^*$$

sumando menos la primera más la segunda se tiene:

$$i\hbar\frac{\partial}{\partial t}\left[\psi^*.\psi\right] = \frac{\hbar^2}{2m}\left[\vec{\nabla}.(\psi\vec{\nabla}\psi^*) - \vec{\nabla}.(\psi^*\vec{\nabla}\psi)\right]$$

$$i\hbar\frac{\partial}{\partial t}\left[\psi^*.\psi\right] = -\frac{\hbar^2}{2m}\vec{\nabla}.\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

$$\underbrace{\frac{\partial}{\partial t}\left[\psi^*.\psi\right]}_{\equiv\rho} + \vec{\nabla}.\underbrace{\left[\frac{\hbar}{2mi}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right]}_{\equiv\vec{J}} = 0$$

Acá tenemos una especie de conservación de la probabilidad con:

$$\rho = \Psi^* \Psi = |\Psi| \geq 0 \quad ; \ \rho \text{ es definido positivo}$$

Para la ecuación de Klein-Gordon se tiene:

$$-\frac{\hbar^2}{c^2}\frac{\partial^2 \Psi}{\partial t^2} + \hbar^2 \nabla^2 \Psi = m^2c^2 \Psi \quad y \quad -\frac{\hbar^2}{c^2}\frac{\partial^2 \Psi^*}{\partial t^2} + \hbar^2 \nabla^2 \Psi^* = m^2c^2 \Psi^*$$

, procediendo en modo ídem será:

$$-\frac{\hbar^2}{c^2} \Psi^* \partial_t^2 \Psi + \hbar^2 \Psi^* \nabla^2 \Psi = m^2 c^2 \Psi^* \Psi$$

$$-\frac{\hbar^2}{c^2} \Psi\, \partial_t^2 \Psi^* + \hbar^2 \Psi\, \nabla^2 \Psi^* = m^2 c^2 \Psi . \Psi^*$$

$$\frac{\hbar^2}{c^2}\left[\Psi \partial_t^2 \Psi^* - \Psi^* \partial_t^2 \Psi\right] + \hbar^2 \left[\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*\right] = 0$$

$$\frac{\cancel{\hbar^2}}{c^2}\left[\qquad\qquad\right] + \cancel{\hbar^2}\vec{\nabla}.\left[\Psi \vec{\nabla}\Psi^* - \Psi^* \vec{\nabla}\Psi\right] = 0$$

$$\frac{\hbar}{c^2 2mi}\frac{\partial}{\partial t}\left[\Psi \partial_t \Psi^* - \Psi^* \partial_t \Psi\right] - \vec{\nabla}.\underbrace{\left[\frac{\hbar}{2mi}\left(\Psi^* \vec{\nabla}\Psi - \Psi \vec{\nabla}\Psi^*\right)\right]}_{\equiv \vec{J}} = 0$$

Como

$$\begin{cases} \partial_t(\Psi . \partial_t \Psi^*) = \partial_t \Psi . \partial_t \Psi^* + \Psi . \partial_t^2 \Psi^* \\ \partial_t(\Psi^* \partial_t \Psi) = \partial_t \Psi^* \partial_t \Psi + \Psi^* \partial_t^2 \Psi \end{cases} \Rightarrow$$

$$\frac{\partial}{\partial t}\left[\frac{\hbar}{2mc^2 i}\left(\Psi \partial_t \Psi^* + \Psi^* \partial_t \Psi\right)\right] + \vec{\nabla}.\left[\frac{\hbar}{2mi}\left(\Psi^* \vec{\nabla}\Psi - \Psi \vec{\nabla}\Psi^*\right)\right] = 0$$

Pero ahora resulta que $\rho \equiv \frac{\hbar}{2mc^2 i}\left(\Psi^* \partial_t \Psi - \Psi\, \partial_t \Psi^*\right)$, lo cual no es definido positivo.

Veamos un ejemplo; la partícula libre es una onda viajera.

$$\Psi = e^{i(\vec{k}.\vec{x} - \omega t)} = e^{\frac{i}{\hbar}(\vec{p}.\vec{x} - E.t)} \Rightarrow$$

$$\rho = \frac{\hbar}{2mc^2 i}\left(e^{-\frac{i}{\hbar}(p.x - E.t)}\, e^{\frac{i}{\hbar}(px - Et)}\left(-\frac{iE}{\hbar}\right) - e^{\frac{i}{\hbar}(px - Et)}\, e^{-\frac{i}{\hbar}(p.x - Et)} . \frac{iE}{\hbar}\right)$$

$$\begin{array}{|l} E = h.\nu \\ \ \ = \frac{2\pi}{2\pi} h.\nu \\ E = \omega.\hbar \end{array}$$

Recordar

$$\rho = \frac{\cancel{\hbar}}{\cancel{2}mc^2 \cancel{i}}\left[-\frac{\cancel{2i}E}{\cancel{\hbar}}\right] = -\frac{E}{mc^2}$$

y si $E > 0 \rightarrow \rho < 0$ lo cual no está muy católico

$\downarrow$ pues una partícula libre tiene $p^2/2m > 0$

En el caso de Schrödinger es:

$$\rho = \Psi^* . \Psi = 1 \rightarrow \text{es la normalización}$$

Como se ve, Klein-Gordon presenta el problema de densidades ρ de probabilidad que pueden no ser positivas

3. Sean: $\vec{J} = \vec{x}\times\vec{p}\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix}\vec{\sigma}&0\\0&\vec{\sigma}\end{pmatrix}$ y el hamiltoniano relativista para una partícula de spin 1/2 es el de Dirac

$$\hat{H} = \vec{\alpha}\cdot\vec{p} + \beta m$$

$$\hat{H} = \begin{pmatrix}0&\vec{\sigma}\\\vec{\sigma}&0\end{pmatrix}\cdot\vec{p}\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix} + \begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}m$$

$$\hat{H} = \begin{pmatrix}0&\sigma_x\\\sigma_x&0\end{pmatrix}\begin{pmatrix}p_x&0\\0&p_x\end{pmatrix} + \begin{pmatrix}0&\sigma_y\\\sigma_y&0\end{pmatrix}\begin{pmatrix}p_y&0\\0&p_y\end{pmatrix} + \begin{pmatrix}0&\sigma_z\\\sigma_z&0\end{pmatrix}\begin{pmatrix}p_z&0\\0&p_z\end{pmatrix} + \begin{pmatrix}m&0\\0&-m\end{pmatrix}$$

ojo que cada escalar es una matriz de 2x2

$$\hat{H} = \sum_{i=1}^{3}\begin{pmatrix}0&\sigma_i p_i\\\sigma_i p_i&0\end{pmatrix} + \begin{pmatrix}m&0\\0&-m\end{pmatrix}$$

$$\vec{J} = \sum_{k=1}^{3}\epsilon_{ijk}\,x_i p_j\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\hat{k} + \frac{\hbar}{2}\sum_{k=1}^{3}\begin{pmatrix}\sigma_k&0\\0&\sigma_k\end{pmatrix}\hat{k}$$

Ahora queremos ver que $[\hat{H},\hat{\vec{J}}]=0 \Rightarrow [\hat{H},\hat{J}_k]=0$ con $k=1,2,3$

Pero ojo porque las σ_k tienen forma diferente para diferente $k \Rightarrow$

$$[H,J_k] = \left[\vec{\sigma}\cdot\vec{p}\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix} + m\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix},\ \epsilon_{ijk}\,x_i p_j\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix} + \frac{\hbar}{2}\sigma_k\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\right]$$

$\equiv B$ $\equiv C$ $\equiv A$ $\equiv D$

Podemos distribuir en cuatro conmutadores

$$*\underline{\underline{A}} = \left[m\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix},\frac{\hbar}{2}\sigma_k\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\right] = m\frac{\hbar}{2}\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\sigma_k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]$$

$$= m\frac{\hbar}{2}\sigma_k\left[\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix},\underbrace{\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}}_{\text{es la identidad}}\right] = 0$$

$$*\underline{\underline{B}} = \left[\sum_{l}\sigma_l p_l\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},\ \epsilon_{ijk}\,x_i p_j\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\right]$$

$$= \left[\sigma_l p_l\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},\ \epsilon_{ijk}\,x_i p_j\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\right] =$$

$$\begin{pmatrix}0&\sigma_l p_l\,\epsilon_{ijk}x_i p_j\\\sigma_l p_l\,\epsilon_{ijk}x_i p_j&0\end{pmatrix} - \begin{pmatrix}0&\epsilon_{ijk}x_i p_j\,\sigma_l p_l\\\epsilon_{ijk}x_i p_j\,\sigma_l p_l&0\end{pmatrix} =$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}[\sigma_l p_l,\epsilon_{ijk}x_i p_j] = \begin{pmatrix}0&1\\1&0\end{pmatrix}\sigma_l[p_l,\epsilon_{ijk}x_i p_j]$$

$$= \begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix}\sigma_l\,\epsilon_{ijk}[p_l,x_i p_j] = \begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix}\sigma_l\,\epsilon_{ijk}\big(x_i\underbrace{[p_l,p_j]}_{=0} + [p_l,x_i]p_j\big)$$

$$= \begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix}\sigma_l\,\epsilon_{ijk}(-i\hbar)\delta_{li}p_j = -\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix}\sigma_i\,\epsilon_{ijk}\,i\hbar\,p_j$$

$$= -\alpha_i\,\epsilon_{ijk}\,i\hbar\,p_j = -i\hbar\,\epsilon_{ijk}\,\alpha_i p_j \Rightarrow -i\hbar(\vec{\alpha}\times\vec{p})_k$$

pero $\hbar=1 \rightarrow$ $-i(\vec{\alpha}\times\vec{p})_k = (B) = [\alpha_i p_i, L_k]$

$* \underline{C}$: $\left[m\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}, \epsilon_{ljk} x_i p_j \begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\right] = m\epsilon_{ijk}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}[1, x_i p_j] = 0$

$* \underline{D}$: $\left[\sigma_l p_l \begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}, \frac{\hbar}{2}\sigma_k\begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}[\sigma_l p_l, \sigma_k] =$

Como: $\left[\sigma_l p_l, \begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\right] = 0$, $\left[\sigma_k, \begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\right] = 0$

$\left[\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}, \begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\right] = 0$

$$= \frac{\hbar}{2}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix} p_l \underbrace{[\sigma_l, \sigma_k]}_{} =$$

$$= \frac{\hbar}{\cancel{2}}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix} p_l \; i\cancel{2}\epsilon_{lkj}\sigma_j$$

$$= i\,\epsilon_{lkj}\,\hbar\, p_l \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}$$

$$= -i\,\epsilon_{ljk}\,\hbar\, p_l \alpha_j$$

$$= -i\hbar\,\epsilon_{ljk} p_l \alpha_j$$

$$= -i\hbar(\vec{p}\times\vec{\alpha})_k$$

con $\hbar = 1 \rightarrow$ $\qquad = i(\vec{\alpha}\times\vec{p})_k$

Juntando todo se tendrá:

$$[H, J_k] = 0 + i(\vec{\alpha}\times\vec{p})_k + 0 + i(\vec{\alpha}\times\vec{p})_k = 0 \quad \Rightarrow \boxed{[H, \vec{J}] = 0}$$

Este resultado es, justamente, la conservación del momento angular total $\vec{J} = \vec{L} + \vec{S}$. EL hamiltoniano para una partícula de spin $1/2$ considerado no tenía ningún potencial que afecte al $\vec{J}$, ahora bien, el hecho de tener spin hace que el $\vec{L}$ por sí solo no se conserve, lo cual está expresado en el conmutador:

$$[\hat{H}, \hat{L}_k] = [\alpha_l p_l + \beta m, \underbrace{(\vec{x}\times\vec{p})_k\begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}}_{\equiv L_k}] = -i(\vec{\alpha}\times\vec{p})_k$$

Asimismo como: $\vec{J} = \vec{L} + \vec{S} \rightarrow \boxed{\hat{\vec{S}} = \frac{\hbar}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}}$ (se identifica el operador de spin (de 4x4).)

y deberá compensar $[\hat{H}, \hat{S}]$ la no-conmutatividad de $[\hat{H}, \hat{L}]$, lo cual sucede:

$$[\hat{H}, \hat{S}_k] = -[\hat{H}, \hat{L}_k] = i(\vec{\alpha}\times\vec{p})_k$$

5.

Para este caso debemos hacer el siguiente reemplazo

$$p^\mu \to p^\mu + eA^\mu \Rightarrow (E,\vec{p}) \to \quad (E,\vec{p}) + e(\underset{\phi}{\underset{\parallel}{A^0}},\vec{A})$$

Comenzamos con:

$$H\Psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\Psi = E\Psi \quad \leftarrow \text{Busco autoestados de la energía}$$

$$\vec{\alpha}\cdot\vec{p}\,\Psi = (E - \beta m)\Psi$$

$$\begin{pmatrix}\vec{\sigma}\cdot\vec{p}\;\Psi_B\\ \vec{\sigma}\cdot\vec{p}\;\Psi_A\end{pmatrix} = \left[\begin{pmatrix}0 & \sigma_1 p_1\\ \sigma_1 p_1 & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_2 p_2\\ \sigma_2 p_2 & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_3 p_3\\ \sigma_3 p_3 & 0\end{pmatrix}\right]\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} = \left[\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix} - \begin{pmatrix}m & 0\\ 0 & -m\end{pmatrix}\right]\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}$$

Resulta un sistemita acoplado $\longrightarrow$

$$\left[\begin{aligned} \vec{\sigma}\cdot\vec{p}\;\Psi_B &= (E-m)\Psi_A \\ \vec{\sigma}\cdot\vec{p}\;\Psi_A &= (E+m)\Psi_B \end{aligned}\right.$$

Ahora, metemos el campo EM con su potencial $A^\mu \to$

$|e\phi| \ll m$

$\left(\frac{e\phi}{m}\right) \ll 1$

$E - m = \mathcal{E} = E_{NR}$

$$\vec{\sigma}\cdot(\vec{p}+e\vec{A})\,\Psi_B = (E + e\phi - m)\Psi_A$$

$$\vec{\sigma}\cdot(\vec{p}+e\vec{A})\,\Psi_A = (E + e\phi + m)\Psi_B$$

$$\begin{cases} (-i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\Psi_B = i\partial_t\Psi_A + (e\phi - m)\Psi_A \\ (-i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\Psi_A = i\partial_t\Psi_B + (e\phi + m)\Psi_B \end{cases}$$

$$(e\phi - m)\Psi_A = -i\partial_t\Psi_A + e\vec{\sigma}\cdot\vec{A}\,\Psi_B - i\vec{\sigma}\cdot\vec{\nabla}\Psi_B$$

$$(e\phi + m)\Psi_B = -i\partial_t\Psi_B + e\vec{\sigma}\cdot\vec{A}\,\Psi_A - i\vec{\sigma}\cdot\vec{\nabla}\Psi_A$$

$$\vec{\sigma}\cdot(i\vec{\nabla} + e\vec{A})\Psi_B = (\mathcal{E}_{NR} + e\phi)\Psi_A$$

$$\vec{\sigma}\cdot(i\vec{\nabla} + e\vec{A})\Psi_A = (\mathcal{E}_{NR} + e\phi + 2m)\Psi_B$$

$$\Psi_A = \frac{\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})}{(\mathcal{E}_{NR}+e\phi)}\cdot\frac{\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})}{\underbrace{(\mathcal{E}_{NR}+e\phi+2m)}_{\sim 2m}}\Psi_A$$

$$(\mathcal{E}_{NR} + e\phi)\Psi_A = \frac{1}{2m}\left(\vec{\sigma}\cdot[i\vec{\nabla} + e\vec{A}]\right)^2\Psi_A$$

usando $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$(\vec{\sigma}\cdot\vec{a})^2 = \vec{a}\cdot\vec{a}$

$$\to \quad [\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})][\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})]$$

$$(i\vec{\nabla} + e\vec{A})^2 + i\,\vec{\sigma}\cdot[(-i\vec{\nabla}+e\vec{A})\times(i\vec{\nabla}+e\vec{A})]$$

$$i\vec{\sigma}\cdot[(e\vec{A}\times i\vec{\nabla}) + (i\vec{\nabla}\times e\vec{A})]$$

$$ie\,\vec{\sigma}\cdot[(\vec{A}\times\vec{P}) + (\vec{P}\times\vec{A})]$$

$$(-i\vec{\nabla}+e\vec{A})^2 + ie\,\vec{\sigma}\cdot(i\,\vec{\nabla}\times\vec{A})$$

$$(\mathcal{E}_{NR} + e\phi)\Psi_A = \frac{1}{2m}\left([-i\vec{\nabla} + e\vec{A}]^2 - e\,\vec{\sigma}\cdot\vec{B}\right)\Psi_A$$

tiramos $e\phi$ en favor de $\mathcal{E}_{NR}$

$$\boxed{\mathcal{E}_{NR}\,\Psi_A = \left(\frac{1}{2m}(i\vec{\nabla} - e\vec{A})^2 - \frac{e}{2m}\,\vec{\sigma}\cdot\vec{B}\right)\Psi_A}$$

Experimentalmente se halla que asociado al $\vec{B}$ se tiene como término correspondiente a la energía magnética del spin del electrón:

$$E_s = -g\,\vec{\mu}\cdot\vec{B}$$

, con

$$\vec{\mu} = \frac{e}{2mc}\vec{S} \quad \Rightarrow \quad E_s = -g\frac{e}{2mc}\vec{S}\cdot\vec{B}$$

entonces con $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ es

$$E_s = -g\cdot\frac{e}{2mc}\frac{\hbar}{2}\vec{\sigma}\cdot\vec{B}$$

en unidades naturales:

$$E_B = -g\cdot\frac{e}{4m}\vec{\sigma}\cdot\vec{B}$$

Pero la ecuación de Dirac da directamente $\frac{e}{2m}\vec{\sigma}\cdot\vec{B}$, con lo cual debe ser $g=2$ (factor giromagnético clásico del electrón)

En los experimentos se vio que: $g \approx 2{,}00232$, con lo cual la ecuación de Dirac tiene bastante confirmación experimental.

6. Sea la ecuación de Dirac escrita en forma covariante:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \qquad \text{con} \quad \gamma^\mu = (\beta, \beta\vec{\alpha})$$
$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) = (\partial_t, \vec{\nabla})$$

Ahora sea $\Psi = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}$ solución de la ecuación de Dirac

$$i\gamma^\mu\partial_\mu \Psi = i\left(\beta\frac{\partial}{\partial x^0} + \beta\vec{\alpha}\cdot\vec{\nabla}\right)\Psi = i\beta\left(\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\nabla}\right)\Psi = m\Psi$$

La ecuación de Klein-Gordon es:

$$\Box^2\Psi + m^2\Psi = 0 \qquad \text{con} \qquad \Box^2 \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \qquad \partial^\mu = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla}\right) \quad \Rightarrow \quad \partial_\mu\partial^\mu = \frac{\partial^2}{\partial x^0 \partial x_0} - \vec{\nabla}\cdot\vec{\nabla}$$

$$\partial^\mu\partial_\mu = \partial_\mu\partial^\mu \qquad\qquad \partial^\mu\partial_\mu = \frac{\partial^2}{\partial x^0\partial x_0} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial^\mu\partial_\mu = \Box^2$$

$$\gamma^\mu = (\beta, \beta\vec{\alpha}) \qquad\qquad \gamma^\mu\partial_\mu = \beta\cdot\frac{\partial}{\partial x^0} + \beta\vec{\alpha}\cdot\vec{\nabla}$$

$$\begin{pmatrix}\beta & \beta\alpha_1 & \beta\alpha_2 & \beta\alpha_3\end{pmatrix}\begin{pmatrix}\partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3\end{pmatrix} = \beta\frac{\partial}{\partial x^0} + \beta\alpha_1\cdot\frac{\partial}{\partial x^1} + \beta\alpha_2\cdot\frac{\partial}{\partial x^2} + \ldots$$
$$+\beta\alpha_3\frac{\partial}{\partial x^3}$$

Sea $\Psi = (\psi^0, \psi^1, \psi^2, \psi^3)$ solución de la ecuación de Dirac

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \qquad \rightarrow \qquad i\gamma^\mu\partial_\mu\Psi = m\Psi$$

$$\gamma^\nu\partial_\nu(i\gamma^\mu\partial_\mu - m)\Psi = 0$$

$$(i\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu - m\gamma^\nu\partial_\nu)\Psi = 0$$

$$\left(i\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu - m\cdot\frac{m}{i}\right)\Psi = 0$$

$$(\gamma^\nu\gamma^\mu\,\partial_\nu\partial_\mu + m^2)\Psi = \left(\frac{1}{2}[\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu]\partial_\nu\partial_\mu + m^2\right)\Psi = 0$$

$$\left(g^{\nu\mu}\partial_\nu\partial_\mu + m^2\right)\psi = 0$$

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi = \left(\Box^2 + m^2\right)\psi = 0 \rightarrow$$

Hemos llegado a lo siguiente: $\left(\Box^2 + m^2\right)\psi^\mu = 0 \Rightarrow$ c/u de los ψ^μ es solución de la ecuación de Klein-Gordon

Pero No toda solución de Dirac es solución de Klein-Gordon. Esto es más hard de demostrar y no lo haremos.

7.

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0 \Rightarrow$$ Para ver como transforma entre dos frames inerciales hacemos:

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi(x) - m\psi(x) = 0$$

$$i\gamma^\mu \frac{\partial}{\partial x'^\mu}\psi'(x') - m\psi'(x') = 0 \qquad \text{con} \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Proponemos: $\longrightarrow \psi'(x') = S_\Lambda \psi(x) \rightarrow$ con $S_\Lambda \neq S_\Lambda(x)$

$$S_\Lambda^{-1}\psi'(x') = \psi(x)$$

Nota: S_Λ depende de $\Lambda \Rightarrow [S,\Lambda] \neq 0$

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} S_\Lambda^{-1}\psi'(x) - m S_\Lambda^{-1}\psi'(x) = 0$$

$$S_\Lambda i\gamma^\mu \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} S_\Lambda^{-1}\psi'(x) - m S_\Lambda S_\Lambda^{-1}\psi'(x) = 0$$

Auxiliar

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$(\Lambda^\mu{}_\nu)^{-1} x'^\mu = x^\nu$$

$$\Lambda^\nu{}_\mu x'^\mu = x^\nu$$

$$\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\nu}$$

$$\partial_\nu = \partial'_\mu \Lambda^\mu{}_\nu$$

$$\Lambda^\nu{}_\mu \partial_\nu = \partial'_\mu$$

$$\partial_\mu = \frac{\partial}{\partial x'^\nu}\Lambda^\nu{}_\mu$$

$$i\, \underbrace{S_\Lambda \gamma^\mu \Lambda^\nu{}_\mu S_\Lambda^{-1}}\, \frac{\partial}{\partial x'^\nu}\psi'(x') - m\psi'(x') = 0$$

debe ser $\searrow$ $\quad S_\Lambda \gamma^\mu \Lambda^\nu{}_\mu S_\Lambda^{-1} = \gamma^\nu$

$$\boxed{S_\Lambda^{-1}\gamma^\nu S_\Lambda = \Lambda^\nu{}_\mu \gamma^\mu}$$

Se ve que $S_\Lambda^{-1}\gamma^\nu S_\Lambda$ transforma como cuadrivector

$$e^{i\omega^{\mu\nu}S_{\mu\nu}} \quad \text{con} \quad S_{\mu\nu} = -\frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

$$= -\frac{i}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = -\frac{i}{4}\,2\gamma^\mu\gamma^\nu = -\frac{i}{2}\gamma^\mu\gamma^\nu$$

si $\mu \neq \nu$

$$e^{\frac{1}{2}\omega^{\mu\nu}\gamma^\mu\gamma^\nu} = \cos\left(\frac{\omega}{2}\right) + \gamma_\mu\gamma_\nu \operatorname{sen}\left(\frac{\omega}{2}\right)$$

En QM esto era la forma del operador de rotación en SU(2)

$S_\Lambda S_\Lambda^{-1} = 1$

$$\left(1 - \frac{\omega}{2}\gamma^\mu\gamma^\nu\right)\left(1 + \frac{\omega}{2}\gamma^\mu\gamma^\nu\right)$$

$$1 - \frac{\omega^2}{4}\underbrace{\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu}_{\sim 0} = 1$$

con $\mu\nu = 0i$ se tiene

$$e^{\frac{1}{2}\omega_0\gamma^0\gamma^i} = e^{\frac{1}{2}\omega_{0i}\alpha_i} \Rightarrow$$ boost en $\hat{i}$ con v dado por $\omega = 2\tanh\left(\frac{v}{c}\right)$

$$S_{\Lambda\,\text{Boost}} = e^{\frac{1}{2}\omega_{0i}\alpha_i}$$

Con $\mu\nu = ij$ se tiene

$$e^{\frac{1}{2}\omega_{ij}\gamma^i\gamma^j} = e^{\frac{1}{2}\omega_{ij}\alpha_i\alpha_j} = e^{i\omega_{ij}\sigma_k}$$

$$[\alpha_i,\alpha_j] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} \sigma_i\sigma_j - \sigma_j\sigma_i & 0 \\ 0 & \sigma_i\sigma_j - \sigma_j\sigma_i \end{pmatrix}$$

$$= i\begin{pmatrix} 2\sigma_k & 0 \\ 0 & 2\sigma_k \end{pmatrix} = i2\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = 2i\,\sigma_k \mathbb{1}$$

$$S_{\Lambda_{\text{ROTACIÓN}}} = e^{\frac{1}{2}\omega^{ij}\gamma_i\gamma_j}$$

Veamos como transforman algunas cosas.

- $\bar{\Psi}\Psi = \Psi^+\gamma^0\Psi = \Psi^+\gamma^0 S_\Lambda^{-1}S_\Lambda\Psi = \boxed{\bar{\Psi}'\Psi'} \rightarrow$ No se transforma es un invariante de Lorentz

Sabemos que $S_\Lambda\Psi = \Psi' \rightarrow \Psi^+S_\Lambda^+ = \Psi'^+ \rightarrow \Psi^+S_\Lambda^+\gamma^0 = \bar{\Psi}' = \Psi^+\gamma^0 S^{-1}$ $\boxed{\bar{\Psi}\Psi \text{ es un escalar}}$

Uso $S^{-1}\gamma^0 = \gamma^0 S^+ \rightarrow \gamma^0 S^{-1} = S^+\gamma^0$. Ante paridad $\bar{\Psi}\gamma^0\gamma^0\Psi = \bar{\Psi}\Psi$

- $\bar{\Psi}'\gamma^\mu\gamma^5\Psi' = \Psi^+\underbrace{S_\Lambda^+\gamma^0}\gamma^\mu\underbrace{\gamma^5 S_\Lambda}\Psi = \Psi^+\gamma^0 S_\Lambda^{-1}\gamma^\mu S_\Lambda\gamma^5\Psi = \bar{\Psi}S^{-1}\gamma^\mu S\gamma^5\Psi$
$= \bar{\Psi}\Lambda^\mu{}_\nu\gamma^\nu\gamma^5\Psi$ es un cuadrivector

si $\mu\neq 0 \rightarrow \gamma^0\gamma^\mu = -\gamma^\mu\gamma^0$ $\quad S_\Lambda^{-1}\gamma^\nu S_\Lambda = \Lambda^\nu{}_\mu\gamma^\mu$ $\quad [\gamma^5, S_\Lambda] = 0$

$\bar{\Psi}' = \Psi'^+\gamma^0 = \Psi^+S^+\gamma^0$ $\quad S_\Lambda^{-1}\gamma^0 = \gamma^0 S_\Lambda^+$

Para la paridad

$$\bar{\Psi}'\gamma^\mu\gamma^5\Psi' = \bar{\Psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\Psi = \begin{cases} \bar{\Psi}\gamma^5\gamma^0\Psi = -\bar{\Psi}\gamma^0\gamma^5\Psi & \text{si } \mu=0 \\ \bar{\Psi}\gamma^\mu\gamma^5\Psi & \text{si } \mu\neq 0 \end{cases}$$

$\boxed{\bar{\Psi}\gamma^\mu\gamma^5\Psi \text{ transforma como pseudocuadrivector}}$

- $\bar{\Psi}'\gamma^\mu\Psi' = \Psi^+\gamma^0 S_\Lambda^{-1}\gamma^\mu S_\Lambda\Psi = \bar{\Psi}S_\Lambda^{-1}\gamma^\mu S_\Lambda\Psi = \bar{\Psi}\Lambda^\mu{}_\nu\gamma^\nu\Psi$
$= \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\Psi$

$\bar{\Psi}' = \Psi'^+\gamma^0 = (S_\Lambda\Psi)^+\gamma^0 = \Psi^+S_\Lambda^+\gamma^0$
$= \Psi^+\gamma^0\underline{\gamma^0 S_\Lambda^+\gamma^0}$

$\bar{\Psi}' = \bar{\Psi}S_\Lambda^{-1} \Rightarrow \bar{\Psi}'S_\Lambda = \bar{\Psi}$
$S_\Lambda^{-1}\Psi' = \Psi$

$$(\bar{\Psi}'\gamma^\mu\Psi') = \Lambda^\mu{}_\nu(\bar{\Psi}\gamma^\nu\Psi)$$

$$\boxed{J'^\mu \equiv \bar{\Psi}'\gamma^\mu\Psi' = \Lambda^\mu{}_\nu(\bar{\Psi}\gamma^\nu\Psi)}$$

es un cuadrivector

Analicemos la paridad $\quad \Psi' = S_P\Psi = \gamma^0\Psi$
$\Psi'^+ = \Psi^+\gamma^0 \rightarrow \bar{\Psi}' = \bar{\Psi}\gamma^0$

$$\bar{\Psi}'\gamma^\mu\Psi' = \bar{\Psi}\gamma^0\gamma^\mu\gamma^0\Psi = \begin{cases} \bar{\Psi}\gamma^0\Psi & \text{si } \mu=0 \\ -\bar{\Psi}\gamma^\mu\Psi & \text{si } \mu\neq 0 \end{cases} \Rightarrow \boxed{\bar{\Psi}\gamma^\mu\Psi \text{ transforma como Cuadrivector}}$$

$\gamma^\mu\gamma^0 = -\gamma^0\gamma^\mu$

* Analicemos $\Psi^+\Psi$:

$$\Psi^+\Psi \rightarrow \Psi'^+\Psi' = (S_\Lambda\Psi)^+(S_\Lambda\Psi) = \Psi^+S_\Lambda^+S_\Lambda\Psi$$

Se puede ver que $J^0 = \bar{\Psi}\gamma^0\Psi = \Psi^+\gamma^0\gamma^0\Psi = \Psi^+\Psi \Rightarrow \Psi^+\Psi \equiv \rho$ [densidad de probabilidad] es el componente temporal del cuadrivector J^μ $\Rightarrow$ no transforma como cuadrivector.

8.

Las matrices γ^μ son: $\gamma^\mu = (\beta, \beta\alpha_1, \beta\alpha_2, \beta\alpha_3)$

$E>0$ $\quad \Psi(x) = u(0)\, e^{-imt}$
$E<0$ $\quad \Psi(x) = v(0)\, e^{+imt}$

soluciones de $\longrightarrow$ $H\Psi = \beta m \Psi = E\Psi$

Un boost viene dado por: $\exp\left(\omega_{\mu\nu} S^{\mu\nu}\right)$ con $S_{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

en el cual tenemos: $\mu = 0$
$\nu = 1 \rightarrow$ eje del boost $\Rightarrow$

$$e^{-\frac{i}{4}\omega_{01}[\gamma^0,\gamma^1]} = e^{-\frac{1}{4z}\operatorname{atanh}(v)\, 2\, \gamma^0\gamma^1} = e^{-\frac{i}{2}\underbrace{\operatorname{atanh}(v)}_{\vartheta}\gamma^0\gamma^1} = e^{-i\frac{\vartheta}{2}\gamma^0\gamma^1} = \cos\left(\frac{\vartheta}{2}\right) + \gamma^0\gamma^1 \operatorname{sen}\left(\frac{\vartheta}{2}\right)$$

$$[\gamma^0, \gamma^1] = \gamma^0\gamma^1 - \gamma^1\gamma^0 = \gamma^0\gamma^1 + \gamma^0\gamma^1 = 2\gamma^0\gamma^1$$

$$\gamma^0\gamma^1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix} \rightarrow$$

$$\Psi'_{Boost}(x) = e^{i\frac{\vartheta}{2}\alpha_1}\Psi(x) = e^{i\frac{\vartheta}{2}\alpha_1}\begin{pmatrix} 1 \\ 0\end{pmatrix} \cdot e^{-imt}$$

Habría que considerar la $\sum$ que representa a e^{algo} y evaluar términos genéricos. $\longrightarrow$

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_1\end{pmatrix}$$
$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix} 0 \\ \sigma_1\end{pmatrix} = \begin{pmatrix} \sigma_1^2 \\ 0\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$$
$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix} \sigma_1^2 \\ 0\end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_1^3\end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_1\end{pmatrix}$$
$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix} 0 \\ \sigma_1^3\end{pmatrix} = \begin{pmatrix} \sigma_1^4 \\ 0\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix}^n\begin{pmatrix} 1 \\ 0\end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ 0\end{pmatrix} & n \text{ par} \\ \begin{pmatrix} 0 \\ \sigma_1\end{pmatrix} & n \text{ impar}\end{cases}$$

...

Aquí ya está dada la recurrencia, la cual generará un seno y un coseno $\rightarrow$

$$\Psi'_{Boost}(x) = \left[\begin{pmatrix} 1 \\ 0\end{pmatrix}\cos\left(\frac{\vartheta}{2}\right) + \begin{pmatrix} 0 \\ \sigma_1\end{pmatrix}\operatorname{sen}\left(\frac{\vartheta}{2}\right)\right] e^{-imt}$$

$$\boxed{\begin{aligned} \Psi'_{Boost}(x) &= \left[\chi_+ \cos\left(\frac{\vartheta}{2}\right) + \chi_- \sigma_1 \operatorname{sen}\left(\frac{\vartheta}{2}\right)\right] e^{-imt}\, u(0) \\ \Psi'_{Boost}(x) &= \left[\chi_+ \cos\left(\frac{\vartheta}{2}\right) + \chi_- \sigma_1 \operatorname{sen}\left(\frac{\vartheta}{2}\right)\right] e^{+imt}\, v(0)\end{aligned}}$$

Soluciones de tipo onda plana

donde $\chi_+ = \begin{pmatrix} 1 \\ 0\end{pmatrix}$ $\quad \chi_- = \begin{pmatrix} 0 \\ 1\end{pmatrix}$

9.

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) \quad \text{con} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow$$

- $P_+ + P_- = \frac{1}{2} + \frac{\gamma^5}{2} + \frac{1}{2} - \frac{\gamma^5}{2} = \frac{1}{2} + \frac{1}{2} = 1$

- $$\begin{aligned} P_\pm^2 &= \frac{1}{2}(1 \pm \gamma^5)\frac{1}{2}(1 \pm \gamma^5) \\ &= \frac{1}{4}(1 \pm \gamma^5 \pm \gamma^5 + \gamma^5\gamma^5) \\ &= \frac{1}{4}(1 \pm 2\gamma^5 + (\gamma^5)^2) = \frac{1}{4}(1 \pm 2\gamma^5 + 1) = \frac{1}{2}(1 \pm \gamma^5) = P_\pm \end{aligned}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\, i\gamma^0\gamma^1\gamma^2\gamma^3 = -1\, \gamma^{0^2}\gamma^{1^2}\gamma^{2^2}\gamma^{3^2} = -1.1.1.1$$

- $$\begin{aligned} P_+ P_- &= \frac{1}{2}(1 + \gamma^5)\frac{1}{2}(1 - \gamma^5) \\ &= \frac{1}{4}[1 + \cancel{\gamma^5} - \cancel{\gamma^5} - (\gamma^5)^2] = 0 \end{aligned}$$

- $$\begin{aligned} \gamma^5 P_\pm \psi &= \gamma^5 \frac{1}{2}(1 \pm \gamma^5)\psi = \frac{1}{2}(\gamma^5 \pm (\gamma^5)^2)\psi = \frac{1}{2}(\pm 1 + \gamma^5)\psi \\ &= \pm \frac{1}{2}(1 \pm \gamma^5)\psi = \pm P_\pm \psi \end{aligned}$$

Estos operadores $P_\pm$ son proyectores

Verificaremos el límite ultrarelativista para una partícula libre.

$$E^2 = p^2 + m^2 \qquad p \gg m$$

$$E = \sqrt{1 + \left[\frac{m}{p}\right]^2} \cdot p \qquad 1 \gg \frac{m}{p}$$

$$E \approx p\left(1 + \frac{1}{2}\frac{m^2}{p^2}\right)$$

$$E \approx p + \frac{m^2}{2p}$$

$$E \approx p$$

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

$$H\psi = \vec{\alpha}\cdot\vec{p}\,\psi + \beta m \psi$$

$$H\psi = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\psi + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}\psi$$

$$E\psi = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\psi + m\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\psi$$

$$E\psi \cong \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\psi \Rightarrow$$

$$\psi \approx \begin{pmatrix} 0 & \frac{\vec{\sigma}\cdot\vec{p}}{E} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E} & 0 \end{pmatrix}\psi$$

$$\gamma^5\psi \simeq \gamma^5\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\frac{\vec{p}}{E}\psi$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\beta\beta\alpha_1\beta\alpha_2\beta\alpha_3 = i\alpha_1\beta\alpha_2\beta\alpha_3 =$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

las matrices α_i, β anticonmutan todas

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\{\beta, \alpha_i\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} = 0$$

$$\beta . \alpha_i = -\alpha_i \beta$$

$$\gamma^5 = i\,\alpha_1 \beta \alpha_2 \beta \alpha_3 = -i\,\alpha_1 \beta \beta \alpha_2 \alpha_3 = -i\,\alpha_1 \alpha_2 \alpha_3$$

$$\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1\sigma_2\sigma_3 \\ \sigma_1\sigma_2\sigma_3 & 0 \end{pmatrix} =$$

$$\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i\,\mathbb{1}_{2\times 2} \Rightarrow$$

$$\alpha_1\alpha_2\alpha_3 = i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Rightarrow \gamma^5 = -i\,i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

, con lo cual, $\gamma^5 \vec{\alpha} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \Rightarrow$

$$\gamma^5 \psi \approx \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \cdot \frac{\vec{p}}{E}\psi$$

pero $E^2 = p^2 + m^2$ y en el límite ultrarelativista $p \gg m$

$E \approx p$

$$\boxed{\gamma^5 \psi \approx \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \cdot \frac{\vec{p}}{|\vec{p}|}\psi}$$