GUÍA 8: Propiedades Térmicas

1.

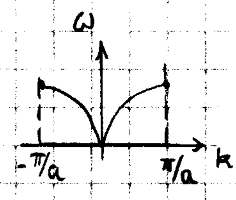
Para la cadena lineal (1D) monoatómica teníamos

$$\omega(k) = \sqrt{\frac{2K}{M}\left(1 - \cos[ka]\right)}$$

$$\omega(k) = \sqrt{\frac{2K}{M} \cdot 2\,\text{sen}\left(\frac{ka}{2}\right)} = 2\sqrt{\frac{K}{M}} \cdot \text{sen}\left(\frac{ka}{2}\right)$$

$$\frac{d\omega(k)}{dk} = \cancel{2}\sqrt{\frac{K}{M}} \cdot \cos\left(\frac{ka}{2}\right) \cdot \frac{a}{\cancel{2}}$$

En 1D era



$$\omega_0 \equiv \omega_m = 2\sqrt{\frac{K}{M}}$$

$$\omega^2 = \omega_0^2\, \text{sen}^2(ka/2)$$
$$\omega^2 - \omega_0^2 = -\cos^2(ka/2)\,\omega_0^2$$
$$\omega_0^2 \cos^2(ka/2) = \omega_0^2 - \omega^2$$

$$\omega(k) = \omega_0 \cdot \text{sen}\left(\frac{ka}{2}\right)$$

$$\frac{d\omega}{dk} = \omega_0 \cdot \cos\left(\frac{ka}{2}\right)\frac{a}{2}$$

$$g(\omega) = \sum_j \int \frac{d\vec{k}}{(2\pi)}\,\delta(\omega - \omega_j(\vec{k})) = \sum_j \int_0^{\pi/a} \frac{\cancel{2}\,dk}{\cancel{2}\pi}\,\delta(\omega - \omega_j(k))$$

$$dk = \frac{2 \cdot d\omega}{a\,\omega_0 \cdot \cos\left(\frac{ka}{2}\right)}$$

$$dk = \frac{2\,d\omega}{a\sqrt{\omega_0^2 - \omega^2}}$$

$$g(\omega) = \sum_j \int_0^{\omega_0} \frac{2 \cdot d\omega}{\pi a \sqrt{\omega_0^2 - \omega^2}}\,\delta(\omega - \omega_0)$$

si $\omega < \omega_0 \rightarrow$

$$\boxed{g(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}}}$$

* Otro Modo

$$\frac{dn}{d\omega} = g(\omega) = \underbrace{\frac{dn}{dk}}_{\text{del ejercicio siguiente}} \cdot \frac{dk}{d\omega} = \boxed{\frac{1}{\pi} \cdot \frac{2}{a[\omega_0^2 - \omega^2]^{1/2}} = g(\omega)}$$

2. Rama óptica sólido 3D, cerca de $k=0$

$$\omega(k) = \omega_0 - Ak^2$$

$$k = \sqrt{\frac{\omega_0 - \omega}{A}}$$

$$\frac{d\omega}{dk} = -2Ak = -2A\,\frac{\sqrt{\omega_0-\omega}}{A^{1/2}} = +2A^{1/2}(\omega_0-\omega)^{1/2}$$

$$g(\omega) = \sum_J \int \frac{d\vec{k}}{(2\pi)^3}\,\delta(\omega - \omega[\vec{k}])$$

$$g(\omega) = \sum_J \iint \frac{k^2\,dk\,d\Omega}{8\pi^3}\,\delta(\omega-\omega[\vec{k}])$$

$$g(\omega) = \sum_J \frac{1}{2\pi^2}\int k^2 dk\,\delta(\omega-\omega[k])$$

Considero $g_j(\omega) \Rightarrow$ $$g_j(\omega) = \frac{1}{2\pi^2}\int \frac{[\omega_0-\omega]}{A}\,\frac{d\omega}{[2A^{1/2}(\omega_0-\omega)^{1/2}]}\,\delta(\omega-\omega[k])$$

$\downarrow$ con ω

$$g_j(\omega) = \frac{1}{2\pi^2}\cdot\frac{A^{-3/2}}{(2)}(\omega_0-\omega)^{1/2}$$

$$g(\omega) = \left(\frac{1}{2\pi}\right)^3 2A^{-3/2}\pi\,(\omega_0-\omega)^{1/2}$$

* Otro Método:

$$N = \left(\frac{L}{2\pi}\right)^3 \frac{4}{3}\pi k^3 \Rightarrow \quad n = \frac{k^3}{6\pi^2}$$

$$\frac{dN}{dk} = \left(\frac{L}{2\pi}\right)^3 4\pi k^2 \quad ; \quad \frac{dk}{d\omega} = \frac{1}{2A^{1/2}(\omega_0-\omega)^{1/2}}$$

$$g(\omega) = \left(\frac{L}{2\pi}\right)^3 4\pi\left(\frac{\omega_0-\omega}{A}\right)\frac{1}{2.A^{1/2}(\omega_0-\omega)^{1/2}}$$

$$\boxed{g(\omega) = \left(\frac{L}{2\pi}\right)^3 2\pi A^{-3/2}[\omega_0-\omega]^{1/2}}$$ con $\omega<\omega_0$

* Cálculo de Expresiones generales para $g(\omega)$

$$N = \sum_k f_{BE}$$

$$N = \int \frac{d\vec{k}}{(2\pi)^d} L^d . f_{BE}$$

$$\frac{N}{L^d} = n = \int \frac{d\vec{k}}{(2\pi)^d} f_{BE}$$

* 1D

$$n = \int \frac{2dk}{\cancel{2}\pi} f_{BE}$$

$$\pi n = k$$

* 2D

$$n = \int \frac{k\,dk\,d\Omega}{(2\pi)^2} f_{BE}$$

$$n = \frac{\cancel{2\pi}}{{}_2\cancel{4}\pi^2}\frac{k^2}{2}$$

$$4\pi n = k^2$$

* 3D

$$n = \int \frac{k^2\,dk\,d\Omega}{(2\pi)^3} f_{BE}$$

$$n = \frac{\cancel{4\pi}}{{}_2\cancel{8}\pi^{\cancel{3}2}}\frac{k^3}{3}$$

$$6\pi^2 n = k^3$$

4.

Medio elástico continuo $\rightarrow \quad \omega(k) = c\,k$

$$E = \sum_j \sum_k \hbar\omega_j^k \left(n_j^k + \frac{1}{2}\right)$$

$$E = \sum_j \sum_k \hbar c k \left(\frac{1}{e^{\beta\hbar c k} - 1} + \frac{1}{2}\right)$$

Como me interesa el $c_v = \frac{d\epsilon}{dT}$; me quedo con lo que depende de la T

$$E' = \sum_j \sum_k \frac{\hbar c k}{e^{\beta\hbar c k} - 1}$$

$*\ \beta\hbar c k = \frac{\hbar c k}{kT} \ll 1 \rightarrow$ alta T $\rightarrow \underbrace{\sum_k}^{k\text{'s permitidos}}$, $\frac{1}{e^x - 1} \approx \frac{1}{x}\left[1 - \frac{x^2}{2}\right]$

$*\ \beta\hbar c k = \frac{\hbar c k}{kT} \gg 1 \rightarrow$ baja T $\rightarrow \sum_k \rightarrow \int d\vec{k} \left(\frac{L}{2\pi}\right)^d$

- Alta T ; 1D

$$E' = \sum_j \sum_k \cancel{\hbar c k}\left(\frac{1}{\cancel{\hbar c k}} k_B T\right) = k_B T \overbrace{\sum_j}^{1 \text{ rama en } 1D} \sum_k = N k_B T$$

$$\epsilon' = \frac{E'}{L} = \frac{N}{L} k_B T$$

$$\boxed{C_v = \frac{N}{L} k_B}$$

- baja T ; 1D

$$E' = \sum_j \sum_k \frac{\hbar c k}{e^{\frac{\hbar c k}{k_B T}} - 1} \qquad \int d\vec{k} \rightarrow \int dk\, 2$$

$$E' = \underbrace{\sum_j}_{1} \frac{L}{2\pi} \int_0^\infty \frac{2\,dk\,\hbar c k}{e^{\beta\hbar c k} - 1} \qquad \begin{matrix}\beta\hbar c k = x \\ \hbar c\, dk = kT\, dx\end{matrix}$$

$$E' = \frac{L}{\cancel{2}\pi} \cancel{2} \frac{1}{\hbar c} \int_0^\infty \frac{dx\,(kT)^2 x}{(e^x - 1)}$$

$$\epsilon' = \frac{k_B^2 T^2}{(\hbar c)\pi} \int_0^\infty \frac{x\,.dx}{e^x - 1} = \frac{k_B^2 T^2}{\hbar c\, \cancel{\pi}} \frac{\pi^{\cancel{2}}}{6} \rightarrow \boxed{C_v = \frac{k_B^2 \pi T}{3\hbar c}}$$

• Alta T; 2D

$$E' = \sum_j \sum_k \hbar c k \left(\frac{k_B T}{\hbar c k}\right) = k_B T \sum_j \sum_k = k_B T \; (2N)$$

$$\epsilon' = 2\frac{N}{L^2} k_B T \rightarrow$$

$$\boxed{C_v = 2 n k_B}$$

• Baja T; 2D

$$E' = \sum_j \sum_k \frac{\hbar c k}{e^{\hbar c k / k_B T} - 1}$$

$$\sum_k \rightarrow \int d\vec{k} = \iint d\Omega \, k \, dk \left(\frac{L}{2\pi}\right)^2$$

$$E' = \sum_j \frac{L^2}{\cancel{4}\pi^{\cancel{2}}} \int_0^\infty \frac{\cancel{4\pi} \hbar c k^2 \, dk}{e^{\hbar c k \beta} - 1}$$

$$\frac{\hbar c k}{k_B T} = x \rightarrow \hbar c \, dk = k_B T \, dx$$

$$\epsilon' = \frac{2}{\pi} \int_0^\infty \frac{k_B T \, dx \, (k_B T / \hbar c)^2 x^2}{e^x - 1}$$

$$\epsilon' = \frac{2}{\pi} \frac{k_B^3 T^3}{\hbar^2 c^2} \int_0^\infty \frac{dx \cdot x^2}{e^x - 1}$$

$$\Gamma(3)\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \cdots\right) = \frac{\pi^3}{\text{algo}}$$

$$2!$$

$$C_v = \frac{6}{\pi} \frac{k_B^3 T^2}{\hbar^2 c^2} \underbrace{\int_0^\infty \frac{dx \cdot x^2}{e^x - 1}}_{\Phi}$$

$$\boxed{C_v = \frac{6\Phi}{\pi} \frac{k_B^3}{(\hbar c)^2} T^2}$$

3.

Medio elástico continuo $\quad \omega(k) = c.k \quad$ [lineal] $\Rightarrow$

$$g(\omega) = \frac{dn}{d\omega} = \frac{dn}{dk}\cdot\frac{dk}{d\omega} \rightarrow \frac{d}{dk}\left(\frac{k}{\pi}\right)\cdot\frac{d}{d\omega}\left(\frac{\omega}{c}\right) = \frac{1}{\pi}\cdot\frac{1}{c}$$

$$\boxed{g(\omega) = \frac{1}{\pi.c}} \quad \text{en } 1D$$

$$\rightarrow \frac{d}{dk}\left(\frac{k^2}{4\pi}\right)\cdot\frac{d}{d\omega}\left(\frac{\omega}{c}\right) = \frac{k}{2\pi}\cdot\frac{1}{c}$$

$$\boxed{g(\omega) = \frac{\omega}{2\pi c^2}} \quad \text{en } 2D$$

* Por el otro Método:

$$D(\omega) = \sum_j \int \frac{d\vec{k}}{(2\pi)} \delta(\omega - \omega[k])$$

$$D(\omega) = \sum_j \int \frac{\cancel{2}\, dk}{\cancel{2}\pi} \delta(\omega - c.k) \qquad \begin{array}{l}\omega = ck \\ d\omega = c.dk\end{array}$$

$$D_1(\omega) = \sum_j \int \frac{d\omega}{c.\pi} \delta(\omega - ck)$$

$$\boxed{D(\omega) = \frac{1}{c.\pi}} \quad \text{en } 1D$$

$$D(\omega) = \sum_j \int \frac{k\, d\Omega\, dk}{4\pi^2} \delta(\omega - ck) \qquad \begin{array}{l}\omega = ck \\ \frac{d\omega}{c} = dk\end{array}$$

$$D(\omega) = \sum_j \int \frac{\omega}{c} \frac{\cancel{2}\pi}{\cancel{4}_2\pi^2} \frac{d\omega}{c} \delta(\omega - ck)$$

$$\boxed{D(\omega) = \begin{cases} \dfrac{\omega}{\pi c^2} & \text{si} \quad \omega \leq \omega_M = c.k_M \\ 0 & \text{si} \quad \omega > \omega_M = c.k_M \end{cases}} \quad \text{en } 2D$$

Por supuesto $D(\omega) = \sum_j g_j(\omega)$ donde j es la rama. En nuestras aproximaciones consideramos

$$g_j(\omega) = g(\omega) \quad \forall\, J = d \text{ (dimensión)} \Rightarrow$$

$$\text{en } 2D, \text{ es: } D(\omega) = 2\, g(\omega)$$

5.

EL Modelo de Debye propone $\omega = c.k$ ← para todas las frecuencias

$$E' = \sum_j \sum_k \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$E' = \sum_j \int_{\text{hasta } k_D} d\vec{k} \left(\frac{\hbar c k}{e^{\beta\hbar c k}-1}\right) \frac{L^d}{(2\pi)^d}$$

* 3D

$$\frac{L^3}{8\pi^3}\int_0^{k_D} d\Omega\, dk\, k^2 \frac{\hbar c k}{[e^{\beta\hbar c k}-1]} = \frac{L^3 \cancel{4\pi}}{\cancel{8}_2\pi^{\cancel{3}2}} \int_0^{k_D} \frac{k^3\, \hbar c\, dk}{e^{\beta\hbar c k}-1} = \frac{L^3}{2\pi^2}\int_0^{\beta\hbar c k_D} \frac{(k_B T)^4 dx . x^3}{[e^x-1](\hbar c)^3}$$

$$E' = \sum_j \frac{L^3}{2\pi^2}\frac{(k_B T)^4}{(\hbar c)^3}\int_0^{\frac{\hbar c k_D}{k_B T}} \frac{x^3 . dx}{e^x - 1}$$

$\beta\hbar c k = x$

$\hbar c\, dk = dx\,(k_B T)$

si extendemos la integral hasta $+\infty$

$$E' \approx \sum_j \frac{L^3}{2\pi^2}\frac{k_B^4 T^4}{(\hbar c^3)}\cdot\frac{\pi^4}{15} \rightarrow \epsilon' \approx \frac{\cancel{3}}{2\cancel{\pi^2}}\cdot\frac{k_B^4 T^4}{(\hbar c)^3}\frac{\pi^{\cancel{4}2}}{\cancel{15}_5}$$

$$\boxed{C_v = \frac{\partial\epsilon'}{\partial T} \approx \frac{2}{5}\frac{\pi^2 k_B^4 T^3}{(\hbar c)^3}}$$

$$\frac{2}{5}\pi^2\left(\frac{6\pi^2 n}{k_D^3}\right)\frac{k_B^4 T^3}{(\hbar c)^3} \rightarrow \frac{12}{5}\pi^4 n k_B \frac{T}{\Theta}$$

Sea que no tomamos $k_D \to \infty$ $\Rightarrow$ $\frac{3 . L^3}{2\pi^2}\hbar c \int_0^{k_D}\frac{k^3 . dk}{e^{\beta\hbar c k}-1}$

definimos $k_B\Theta_D = \hbar\omega_D = \hbar k_D c$

$\omega_D = c . k_D$

$$\epsilon' = \frac{3}{2\pi^2}\hbar c\int_0^{k_D}\frac{k^3 dk}{e^{\beta\hbar c k}-1}$$

$$C_v = \frac{\partial}{\partial T}\left(\frac{3\hbar c}{2\pi^2}\int_0^{k_D}\frac{k^3 . dk}{e^{\beta\hbar c k}-1}\right)$$

$\frac{\hbar c k}{k_B T} \equiv x$

$dk = \frac{k_B T}{\hbar c} . dx$

$$C_v = \frac{3\hbar c}{2\pi^2}\int_0^{k_D}\frac{+k^3 . dk . e^{\beta\hbar c k}}{(e^{\beta\hbar c k}-1)^2}\cdot\frac{\hbar c k}{k_B T^2}$$

$$C_v = \frac{3\hbar c}{2\pi^2}\int_0^{\frac{\Theta_D}{T}}\frac{e^x . x^4}{(e^x-1)^2}\left(\frac{k_B T}{\hbar c}\right)^4 \frac{\cancel{k_B T}}{\cancel{\hbar c}} . dx . \frac{\cancel{\hbar c}}{\cancel{k_B} T^2}$$

$$C_v = \frac{3}{2\pi^2}\frac{k_B^4 T^3}{(\hbar c)^3}\int_0^{\frac{\Theta_D}{T}}\frac{x^4 . e^x}{(e^x-1)^2}dx$$

uso → $\boxed{6\pi^2 n = k_D^3}$ en 3D

$$\frac{3}{\cancel{2\pi^2}}(\cancel{6}^3\cancel{\pi^2}n)\frac{1}{k_D^3}\frac{k_B^4 T^3}{(\hbar c)^3}\int$$

$$\boxed{C_v = 9 n k_B\left(\frac{T}{\Theta_D}\right)^3\int_0^{\Theta_D/T}\frac{x^4 e^x}{(e^x-1)^2}dx}$$

HOJA

FECHA

* **2D**

$$E' = \sum_{j} \int_{\text{hasta } k_D} d\vec{k} \left(\frac{\hbar c k}{e^{\beta \hbar c k} - 1} \right) \frac{L^2}{4\pi^2}$$

$$\epsilon' = \frac{1}{2\pi^2} \int_{\theta} \int_{0}^{k_D} \frac{d\Omega \, dk \, k^2 \hbar c}{e^{\beta \hbar c k} - 1} = \frac{1}{\pi} \int_0^{k_D} \frac{\hbar c \, k^2 . dk}{e^{\beta \hbar c k} - 1}$$

$$C_V = \frac{1}{\pi} \int_0^{k_D} \hbar c \, k^2 . dk . \frac{e^{\beta \hbar c k}}{(e^{\beta \hbar c k} - 1)^2} \frac{\hbar c k}{k_B T^2}$$

$$C_V = \frac{1}{\pi} \int_0^{k_D} \frac{(\hbar c)^2 k^3 . dk}{k_B T^2} \frac{e^{\beta \hbar c k}}{(e^{\beta \hbar c k} - 1)^2}$$

$$\frac{\hbar c k}{k_B T} = x$$

$$dk = \frac{k_B T}{\hbar c} dx$$

$$C_V = \frac{1}{\pi} \int_0^{\Theta/T} \frac{(\hbar c)^2}{k_B T^2} \left(\frac{k_B T}{\hbar c} \right)^3 \frac{k_B T}{\hbar c} . \frac{dx \; e^x . x^3}{(e^x - 1)^2}$$

$$\Theta_D = \frac{k_D \hbar c}{k_B}$$

$$C_V = \frac{1}{\pi} \frac{T^2}{(\hbar c)^2} k_B^3 \int_0^{\Theta/T} \frac{dx . x^3 . e^x}{(e^x - 1)^2}$$

uso $\rightarrow$ $\boxed{4\pi n = k_D^2}$

$$\frac{1}{\cancel{\pi}} . \frac{4\cancel{\pi} n}{k_D^2} k_B \frac{k_B^2 T^2}{\hbar^2 c^2} \int$$

$$\boxed{C_V = 4 n \, k_B \left(\frac{T}{\Theta_D} \right)^2 \int_0^{\frac{\Theta}{T}} \frac{dx . x^3 . e^x}{(e^x - 1)^2}}$$

* **1D**

$$\epsilon' = \frac{1}{\cancel{2}\pi} \cancel{2} \int_0^{k_D} dk \frac{k \, \hbar c}{e^{\beta \hbar c k} - 1} = \frac{1}{\pi} \int_0^{k_D} \frac{\hbar c k . dk}{e^{\beta \hbar c k} - 1}$$

$$C_V = \frac{1}{\pi} \int_0^{k_D} \frac{dk . \; k . \hbar c}{(e^{\beta \hbar c k} - 1)^2} . e^{\beta \hbar c k} \frac{\hbar c k}{k_B T^2}$$

$$\frac{\hbar c k}{k_B T} = x$$

$$dk = \frac{k_B T}{\hbar c} dx$$

$$\frac{1}{\pi} \int_0^{\frac{\Theta_D}{T}} \frac{k_B T}{\hbar c} dx \frac{\cancel{(\hbar c)^2}}{\cancel{k_B T^2}} \cancel{\left(\frac{k_B T}{\hbar c} \right)^2} x^2 \frac{e^x}{(e^x - 1)^2}$$

$$\frac{1}{\pi} \frac{k_B^2 T}{\hbar c} \int_0^{\frac{\Theta}{T}} \frac{dx . \; x^2 . e^x}{(e^x - 1)^2}$$

uso $\rightarrow$ $\pi n = k_D$

$$C_V = \frac{n}{k_D} \frac{k_B}{\hbar c} k_B T \int_0^{\frac{\Theta}{T}} \frac{dx . e^x . x^2}{[e^x - 1]^2}$$

$$\boxed{C_V = n \, k_B \left(\frac{T}{\Theta} \right) \int_0^{\frac{\Theta}{T}} \frac{dx . x^2 . e^x}{[e^x - 1]^2}}$$

NOTA

- A bajas T podemos considerar :

$$\int_0^{k_D} dk \rightarrow \int_0^{\infty} dk$$

, dado que el integrando es negligible debido al gran peso de $e^{\frac{\hbar ck}{k_BT}}$

* **3D**:

$$\epsilon' = \frac{3\hbar c}{2\pi^2}\int_0^{\infty}\frac{k^3 dk}{e^{\beta\hbar ck}-1} \qquad \frac{\hbar ck}{k_BT} = x \qquad dk = \frac{k_BT}{\hbar c}dx$$

$$\epsilon' = \frac{3\cancel{\hbar c}}{2\pi^2}\int_0^{\infty}\frac{dx.\,x^3}{e^x-1}\frac{(k_BT)^3}{(\hbar c)^3}\left(\frac{k_BT}{\cancel{\hbar c}}\right)$$

$$\epsilon' = \frac{3}{2\pi^2}\frac{(k_BT)^4}{(\hbar c)^3}\int_0^{\infty}\frac{x^3 dx}{e^x-1}$$

$$\epsilon' = \frac{\cancel{3}^{1}}{2\cancel{\pi^2}}\frac{(k_BT)^4}{(\hbar c)^3}\frac{\cancel{\pi^4}^{2}}{\cancel{15}_{5}} \rightarrow \boxed{C_v \cong \frac{2}{5}\pi^2\frac{k_B^4}{(\hbar c)^3}T^3}$$

* **2D**:

$$\epsilon' = \frac{1}{\pi}\int_0^{\infty}\frac{\hbar c\,k^2 dk}{e^{\beta\hbar ck}-1}$$

$$\epsilon' = \frac{1}{\pi}\int_0^{\infty}k_BT.\,dx\left(\frac{k_BT}{\hbar c}\right)^2\cdot\frac{x^2}{e^x-1}$$

$$C_v = \frac{3k_B^3}{\pi\hbar c}T^2\int_0^{\infty}\frac{x^2.\,dx}{e^x-1} \rightarrow$$

* **1D**:

$$\epsilon' = \frac{1}{\pi}\int_0^{\infty}\frac{k.\,dk.\,\hbar c}{e^{\beta\hbar ck}-1}$$

$$\epsilon' = \frac{1}{\pi}\int_0^{\infty}\left(\frac{k_BT}{\hbar c}\right)^2\frac{dx.\,x\,\hbar c}{e^x-1} \qquad C_v = \frac{1}{\pi}\frac{2k_B^2}{(\hbar c)}T\int_0^{\infty}\frac{dx.\,x}{e^x-1}$$

$$C_v = \frac{\cancel{2}^{1}k_B^2T}{\cancel{\pi}\hbar c}\frac{\cancel{\pi^2}}{\cancel{6}_{3}} \rightarrow \boxed{C_v = \frac{1}{3}\pi\frac{k_B^2T}{(\hbar c)}}$$

- A altas T podemos considerar:

$$\frac{1}{e^x-1} \approx \frac{1}{x}\left[1-\frac{x^2}{2}\right] \quad \text{, dado que } x = \frac{\hbar\omega}{k_BT} \ll 1 \Rightarrow$$

* **3D**:

$$\beta\hbar ck = x$$

$$\epsilon' = \frac{3\hbar c}{2\pi^2}\int_0^{k_D}\frac{k^3dk}{e^{\beta\hbar ck}-1} = \frac{3\cancel{\hbar c}}{2\pi^2}\int_0^{\beta\hbar ck_D}\frac{dx\,x^3}{e^x-1}\left(\frac{k_BT}{\hbar c}\right)^3\left(\frac{k_BT}{\cancel{\hbar c}}\right)$$

$$\epsilon' \cong \frac{3}{2\pi^2}\frac{k_B^4T^4}{(\hbar c)^3}\int_0^{\beta\hbar ck_D}x^2\left(1-\frac{x^2}{2}\right)$$

$$\epsilon' \cong \frac{\cancel{3}}{2\pi^2}\frac{k_B^4T^4}{(\hbar c)^3}\cdot\frac{x^3}{\cancel{3}}\Big|_0^{\beta\hbar ck_D} = \frac{1}{2\pi^2}\frac{k_B^{\cancel{4}}T^{\cancel{4}}}{\cancel{(\hbar c)^3}}\frac{\cancel{\hbar^3c^3}k_D^3}{\cancel{(k_BT)^3}} = \frac{k_BT\,k_D^3}{2\pi^2}$$

$$C_v = \frac{k_Bk_D^3}{2\pi^2} = \frac{\cancel{6}^{3}\cancel{\pi^2}n}{\cancel{2}\cancel{\pi^2}}k_B = \boxed{3nk_B = C_v}$$

Es el calor específico clásico

∗ 2D:

$$\epsilon' = \frac{1}{\pi}\int_0^{k_D} \frac{\hbar c\, k^2 dk}{e^{\beta\hbar c k}-1} = \frac{1}{\pi}\int_0^{\beta\hbar c k_D} \frac{k_B T\, dx\, (k_B T)^2 x^2}{(e^x-1)\,(\hbar c)^2}$$

$\hbar c k = k_B T x$
$\hbar c\, dk = k_B T\, dx$

$$\epsilon' = \frac{(k_B T)^3}{\pi(\hbar c)^2}\int_0^{\beta\hbar c k_D} x\left(1-\frac{x^2}{2}\right) dx$$

$$\epsilon' = \frac{(k_B T)^3}{\pi(\hbar c)^2}\,\frac{x^2}{2}\Big|_0^{\beta\hbar c k_D} = \frac{(k_B T)^{\cancel{3}}\,(\cancel{\hbar c})^{\cancel{2}} k_D^2}{\pi(\cancel{\hbar c})^{2}(\cancel{k_B T})^{2}\, 2} = \frac{k_B T}{2\pi}\, k_D^2$$

$\to 2n$

$$\epsilon' = 2 k_B n T \to \boxed{c_v = 2n k_B}$$

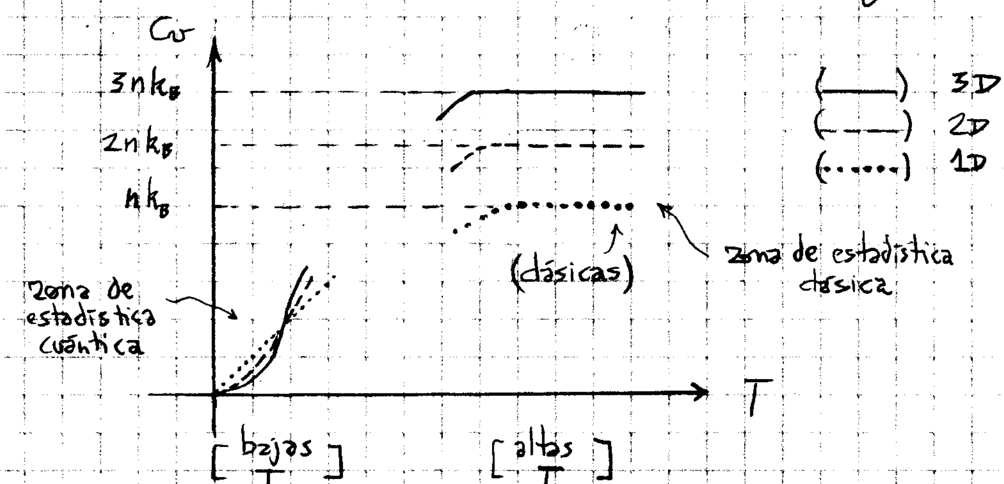
∗ 1D:

$$\epsilon' = \frac{1}{\pi}\int_0^{k_D}\frac{k\, dk\, \hbar c}{e^{\beta\hbar c k}-1} = \frac{1}{\pi}\int_0^{\beta\hbar c k_D}\frac{(k_B T)^2 x\, dx\, \hbar c}{(\hbar c)^2 (e^x-1)} = \frac{1}{\pi}\frac{(k_B T)^2}{\hbar c}\int_0^{\beta\hbar c k_D}\frac{x\, dx}{e^x-1}$$

$$\epsilon' = \frac{1}{\pi}\frac{(k_B T)^2}{\hbar c}\int_0^{\beta\hbar c k_D} dx = \frac{1}{\pi}\frac{(k_B T)^{\cancel{2}}}{\cancel{\hbar c}}\,\frac{\cancel{\hbar c} k_D}{\cancel{k_B T}} = \frac{k_B T\, k_D}{\pi}$$

$$c_v = \frac{k_B k_D}{\pi} = \boxed{n k_B = c_v}$$

De los análisis de Debye tenemos cubiertas dos regiones, $T \sim 0$ y T altas



∗ 3D: Aporte de Einstein

supongo tres ramas ópticas

$$E = \sum_{j}\sum_{k} \frac{\hbar\omega_E}{e^{\beta\hbar\omega_E}-1} = \frac{3N\,\hbar\omega_E}{e^{\beta\hbar\omega_E}-1} \to \underset{(3\text{ ópticas})}{\epsilon'} = \frac{3n\,\hbar\omega_E}{e^{\beta\hbar\omega_E}-1}$$

ramas ópticas

Con T baja es muy pequeña ϵ', pues $e^{\beta\hbar\omega_E} \to \infty$

Con T alta se tiene $\to$ $\epsilon' = 3n\cancel{\hbar\omega_E}\,\frac{k_B T}{\cancel{\hbar\omega_E}}\left(1 - \frac{1}{2}\left[\frac{\hbar\omega_E}{k_B T}\right]^2\right)$

$$\epsilon' = 3n k_B T - \frac{3n(\hbar\omega_E)^2}{k_B T}$$

$$c_{v\,\text{óptico}} = 3n k_B + \frac{3n(\hbar\omega_E)^2}{k_B T^2}$$

Por el lado de la densidad de estados tendremos:

en **3D**: $\cdot\, g(\omega) = \frac{dn}{dk}\cdot\frac{dk}{d\omega} = \frac{d}{dk}\left(k^3 \frac{1}{6\pi^2}\right)\cdot\frac{d}{d\omega}\left(\frac{\omega}{c}\right)$ $\left(\frac{L}{2\pi}\right)$

$$g(\omega) = \frac{k^2}{2\pi^2}\cdot\frac{1}{c} \qquad (\text{por banda})$$

$$D(\omega) = \sum_j \int \frac{d\vec{k}}{(2\pi)^3}\,\delta(\omega - ck)$$

$$3\cdot\frac{\overset{1}{4}\pi}{\underset{2}{8}\pi^3}\int_0^{k_D} k^2 dk\ \delta(\omega - ck) \qquad \begin{aligned}\omega &= ck\\ d\omega &= c\,dk\end{aligned}$$

$$D(\omega) = \frac{3}{2\pi^2}\int_0^{\omega_D/c}\frac{d\omega}{c}\cdot\frac{\omega^2}{c^2}\,\delta(\omega - ck)$$

$$= \frac{3}{2\pi^2}\,\frac{\omega^2}{c^3} \qquad \text{si} \quad \omega \le \omega_D = ck_D$$

$$\boxed{D(\omega) = \begin{cases}\dfrac{3k^2}{2\pi^2 c} & \text{si} \quad \omega \le \omega_D = ck_D\\[2mm] 0 & \text{si} \quad \omega > \omega_D = ck_D\end{cases}}$$

$$D(\omega) = \frac{3}{2\pi^2}\cdot\frac{\omega_E^2}{C^3} = n\cdot\delta(\omega - \omega_E)$$