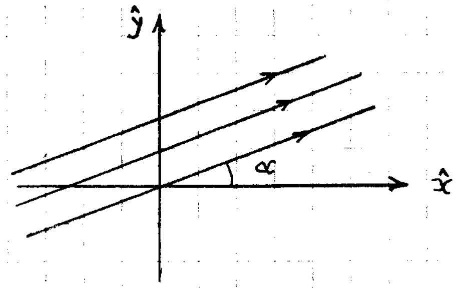
Práctica 5

1. fluidos incompresibles $\rightarrow \rho \neq \rho(\vec{x},t) \rightarrow \mathrm{div}(\vec{v}) = 0 \rightarrow \exists\, \vec{\Psi}:\ \mathrm{rot}(\vec{\Psi}) = \vec{v}.$
invísidos
mov. plano (2D)

$\vec{u}(x,y)$; $\vec{\omega} = \mathrm{rot}(\vec{u})$, $\Psi \equiv A_z(x,y)$

a)



corriente uniforme al infinito

$|\vec{u}| = U_\infty$

$\boxed{\vec{u} = \hat{x}\, U_\infty . \cos\alpha + \hat{y}\, U_\infty . \mathrm{sen}\,\alpha}$

$\boxed{\vec{\omega} = 0 = \mathrm{rot}(\vec{u})}$ (No depende de x,y)

$$\mathrm{rot}(\vec{\Psi}) = \vec{U}$$

$$\frac{\partial\Psi}{\partial y}\hat{x} - \frac{\partial\Psi}{\partial x}\hat{y} = U_\infty . \cos\alpha\, \hat{x} + U_\infty . \mathrm{sen}\,\alpha\, \hat{y}$$

$$\frac{\partial\Psi}{\partial y} = U_\infty . \cos\alpha$$

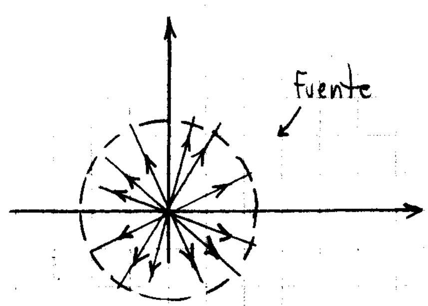
$$\Psi = U_\infty . \cos\alpha . y + C(x)$$

$$-\frac{\partial}{\partial x}\Psi = -\frac{\partial C(x)}{\partial x} = U_\infty . \mathrm{sen}\,\alpha$$

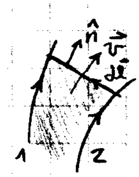
$$C(x) = -U_\infty . \mathrm{sen}\,\alpha . x$$

$$\boxed{\Psi = U_\infty . \cos\alpha . y - U_\infty . \mathrm{sen}\,\alpha . x}$$

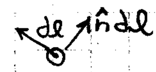
b)



$$Q = \int_1^2 \vec{v} \cdot \hat{n}\, d\ell$$



caudal entre líneas de corriente 1 y 2



La velocidad no tiene componente en $\theta \rightarrow v_\theta = 0$
Por la simetría de rotación es $v_r(r)$

En polares (cilíndricas en $z=0$) es:
$\vec{v} = v\,\hat{r}$ $\quad \hat{n}\,d\ell = d\vec{\ell} \times \hat{z} = r.d\theta\,\hat{r}$

$$\Rightarrow \quad Q = \int_0^{2\pi} \underbrace{v.r.d\theta}_{v\hat{r}\cdot r d\theta \hat{r}} = 2\pi r . v$$

$$\frac{Q}{2\pi r} = v$$

$$\vec{\omega} = \mathrm{rot}(\vec{v}) = \frac{\partial}{\partial z} v_r\,\hat{\theta} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\hat{z} = 0$$

Ahora cambio notación $\vec{v} \equiv \vec{u}$ $\therefore$

$$\boxed{\vec{u} = \frac{Q}{2\pi r}\hat{r}}$$

$$\boxed{\mathrm{rot}(\vec{u}) = \vec{\omega} = 0}$$

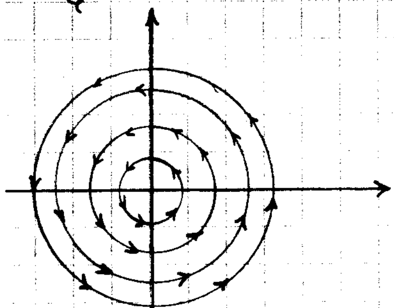
$\mathrm{rot}(\vec{\Psi}) = \vec{u} \Rightarrow$

$$\frac{1}{r}\frac{\partial(\Psi)}{\partial\theta}\hat{r} + -\frac{\partial(\Psi)}{\partial r}\hat{\theta} = \frac{Q}{2\pi r}\hat{r}$$

$$\rightarrow \frac{\partial\Psi}{\partial\theta} = \frac{Q}{2\pi} \qquad \Psi = \frac{Q}{2\pi}\theta$$

$$\boxed{\Psi = \frac{Q}{2\pi}\theta}$$

c) $\Gamma = \oint_{\mathcal{C}} \vec{v} \cdot d\vec{\ell}$



Por consideraciones de simetría

$\vec{u} = u_\theta \hat{\theta} \qquad u_\theta = u_\theta(r) \rightarrow$

$$\Gamma = \int_0^{2\pi} u_\theta \, r . d\theta \rightarrow$$

$$u_\theta = \frac{\Gamma}{2\pi r}$$

$$\boxed{\vec{u} = \frac{\Gamma}{2\pi r} \hat{\theta}}$$

$$\vec{\omega} = rot(\vec{v}) = \frac{1}{r} \frac{\partial}{\partial r}(r . u_\theta) \hat{z}$$

$$\boxed{\vec{\omega} = 0}$$

$$\vec{v} = rot(\Psi \hat{z}) = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \hat{r} - \frac{\partial \Psi}{\partial r} \hat{\theta}$$

$$\frac{\Gamma}{2\pi r} \hat{\theta} = -\frac{\partial \Psi}{\partial r} \hat{\theta} \rightarrow \int \partial \Psi = \int -\frac{\Gamma}{2\pi r} . dr$$

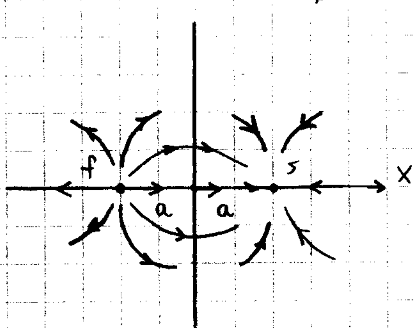
$$\boxed{\Psi = -\frac{\Gamma}{2\pi} . \ln r} \leftarrow \Psi = -\frac{\Gamma}{2\pi} . \ln r$$

d)

$\vec{u}_f = \frac{Q}{2\pi r} \hat{r}$ (fuente) $\qquad \vec{u}_s = -\frac{Q}{2\pi r} \hat{r}$ (sumidero) $\qquad$ (Ambos en el origen)

Habrá que desplazarlos, entonces

$$\vec{u} = \frac{Q}{2\pi} \left(\frac{\vec{r}_f}{r_f^2} - \frac{\vec{r}_s}{r_s^2} \right) = \frac{Q}{2\pi} \left[\frac{(x+a, y)}{\sqrt{(x+a)^2 + y^2}} - \frac{(x-a, y)}{\sqrt{(x-a)^2 + y^2}} \right]$$



$$\vec{u} = \frac{Q}{2\pi} \left[\left(\frac{x+a}{\sqrt{(x+a)^2 + y^2}} + \frac{a-x}{\sqrt{(x-a)^2 + y^2}} \right) \hat{x} + \left(\frac{y}{\sqrt{(x+a)^2 + y^2}} - \frac{y}{\sqrt{(x-a)^2 + y^2}} \right) \hat{y} \right]$$

Pero esta expresión es inmanejable para tomar el límite $Q.a \rightarrow$ constante si $Q \rightarrow \infty$, $a \rightarrow 0$

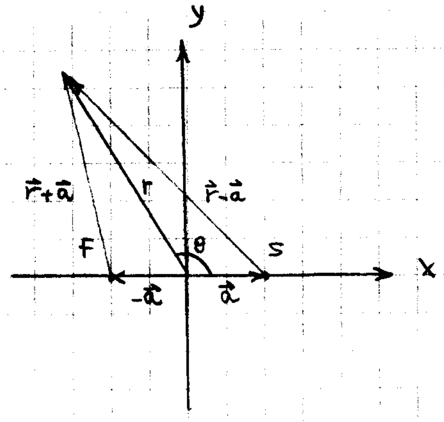
$$\vec{u}(0,0) = \frac{Q}{2\pi} \left[\frac{2a}{\sqrt{a^2+y^2}} \hat{x} + 0 \hat{y} \right]$$

Es conveniente usar el potencial ϕ, amparados en la irrotacionalidad del flujo para fuente/sumidero $\rightarrow$

$$\phi = \frac{Q}{2\pi} . \ln r \rightarrow \phi = \frac{Q}{2\pi} \left[\ln |\vec{r} + \vec{a}| - \ln |\vec{r} - \vec{a}| \right]$$

, con $\vec{r} = r\hat{r} = r(\cos\theta \hat{x} + \operatorname{sen}\theta \hat{y})$, $\vec{a} = a\hat{x}$

$$\phi = \frac{Q}{2\pi} \left(\ln r \left| \hat{r} + \frac{\vec{a}}{r} \right| - \ln r \left| \hat{r} - \frac{\vec{a}}{r} \right| \right)$$



$$|\vec r - \vec a| = \sqrt{r^2+a^2-2ra\cdot\cos\theta} = r\sqrt{1+a^2/r^2 - 2\cos\theta\, a/r}$$
$$|\vec r + \vec a| = \sqrt{r^2+a^2+2ra\cdot\cos\theta} = r\sqrt{1+a^2/r^2 + 2\cos\theta\, a/r}$$

tiramos orden 2 →

$$|\vec r - \vec a| = r\left(1-2\cos\theta\cdot\frac{a}{r}\right)^{1/2}$$
$$= r\left(1+2\cos\theta\cdot\frac{a}{r}\right)$$

$$\ln\left(\frac{|\vec r+\vec a|}{|\vec r-\vec a|}\right) \cong \frac{1}{2}\cdot\ln\left(\frac{1+\cos\theta\cdot a/r}{1-\cos\theta\cdot a/r}\right) \cong \ln\ \left(1+\cos\theta\frac{a}{r}\right)\left(1+\cos\theta\cdot\frac{a}{2r}\right)$$

$$= \frac{1}{2}\left[\ln\left(1+\frac{2a}{r}\cdot\cos\theta\right) - \ln\left(1-\frac{2a}{r}\cdot\cos\theta\right)\right]$$

Taylor para $\underset{\text{orden } 1}{\ln(1+\varepsilon)} = \frac{1}{1+\varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon = \varepsilon$, con $\varepsilon \ll 1$

$$\cong \frac{1}{2}\left(\frac{2a}{r}\cos\theta - -\frac{2a}{r}\cdot\cos\theta\right)$$

$$= 2\frac{a}{r}\cdot\cos\theta$$

$$\phi = \frac{Q}{2\pi}\cdot\frac{2a}{r}\cdot\cos\theta$$

Ahora tomamos el límite con $Q2a \to p$, $Q\to\infty$, $a\to 0$ ⇒

$$\boxed{\phi = \frac{p}{2\pi r}\cdot\cos\theta}$$

Potencial de velocidad de un dipolo

$$\phi = \frac{\vec p\cdot\vec r}{2\pi r^2}$$

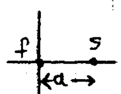
Sea el caso sencillo

$$\phi = \frac{Q}{2\pi}\left(\ln|\vec r| - \ln|\vec r - \vec a|\right)$$

$$\phi = -\frac{Q}{2\pi}\cdot\ln\left|\frac{\vec r-\vec a}{\vec r}\right| = -\frac{Q}{2\pi}\cdot\ln\left(\frac{|\vec r-\vec a|}{r}\right)$$

$$\phi \cong -\frac{Q}{2\pi}\cdot\frac{1}{2}\cdot\ln\left(1-\frac{2a}{r}\cos\theta\right)$$

$a/r \sim 0$ ⇒



$$\phi \cong -\frac{Q}{2\pi}\cdot\frac{1}{\cancel{2}}\left(-\frac{\cancel{2}a}{r}\cdot\cos\theta\right)$$

ahora $Qa \to p$ ⇒

$$\phi = \frac{p}{2\pi r}\cdot\cos\theta$$

Queremos la velocidad $\vec u$ en cilíndricas, que será: $\vec\nabla\phi = \vec u$

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{p\cdot\cos\theta}{2\pi r^2}\ , \quad u_\theta = \frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta} = \frac{-p}{2\pi r^2}\cdot\operatorname{sen}\theta$$

$$\boxed{\vec u = \frac{-p}{2\pi r^2}\left(+\cos\theta\,\hat r + \operatorname{sen}\theta\,\hat\theta\right)}$$

tomar rotor es una operación lineal

$$\operatorname{rot}\vec u = \left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot u_\theta) - \frac{1}{r}\cdot\frac{\partial u_r}{\partial\theta}\right)\hat z$$

(tiene que ser nulo porque una fuente/sumidero tienen rotor nulo)

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\frac{p\cdot\operatorname{sen}\theta}{2\pi r}\right) - \frac{1}{r}\cdot+\frac{p\cdot\operatorname{sen}\theta}{2\pi r^2} = +\frac{1}{r}\cdot\frac{p\cdot\operatorname{sen}\theta}{2\pi r^2} - \frac{1}{r}\cdot\frac{p\cdot\operatorname{sen}\theta}{2\pi r^2}$$

$$\boxed{\vec\omega = 0}$$

además supusimos $\operatorname{rot}(\vec u) = 0$ para usar ϕ

Como tiene rotor nulo puedo derivar una función corriente $\Psi(r,\theta)\hat z$

$$\operatorname{rot}(\Psi\hat z) = \vec u \qquad \frac{1}{r}\cdot\frac{\partial\Psi}{\partial\theta} = u_r \qquad -\frac{\partial\Psi}{\partial r} = u_\theta$$

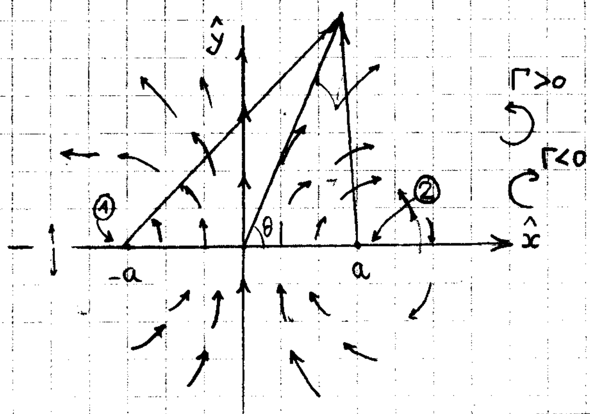
$$\int d\Psi = \int -\frac{p\cos\theta}{2\pi r}\,d\theta \qquad -\frac{p\cdot\operatorname{sen}\theta}{2\pi r^2} + C'(r) = -\frac{p\cdot\operatorname{sen}\theta}{2\pi r^2}$$

$$C'(r) = 0$$

$$\Psi = -\frac{p\cdot\operatorname{sen}\theta}{2\pi r} + C(r)$$

$$\boxed{\vec{\Psi}(r,\theta) = -\frac{P.\,\text{sen}\,\theta}{2\pi r}\,\hat{z}}$$

e)



$\vec{u}_1 = \frac{\Gamma}{2\pi r}\,\hat{\theta}$ vórtice en el origen $r=0$

$$\vec{u} = \frac{\Gamma}{2\pi}\frac{(-\text{sen}\,\theta_1\,\hat{x} + \cos\theta_1\,\hat{y})}{[(x+a)^2+y^2]^{1/2}} - \frac{\Gamma}{2\pi}\frac{(-\text{sen}\,\theta_2\,\hat{x} + \cos\theta_2\,\hat{y})}{[(x-a)^2+y^2]^{1/2}}$$

considero $\Gamma = |\Gamma|$

$$\boxed{\vec{u}(x,y) = \frac{\Gamma}{2\pi}\left[\frac{-y\,\hat{x} + (x+a)\,\hat{y}}{(x+a)^2+y^2} - \frac{-y\,\hat{x} + (x-a)\,\hat{y}}{(x-a)^2+y^2}\right]}$$ dos vórtices

$$\vec{u}(x=0,y) = \frac{\Gamma}{2\pi}\left(\frac{0}{a^2+y^2}\hat{x} + \frac{2a}{a^2+y^2}\hat{y}\right) = \frac{\Gamma a}{\pi(a^2+y^2)}\hat{y} \leftarrow$$ (lo cual parece bastante razonable)

Otra vez aquí convendrá trabajar con el potencial, en este caso conviene usar $\psi \rightarrow$

$\psi_1 = -\frac{\Gamma}{2\pi}\ln r_1$, $\psi_2 = +\frac{\Gamma}{2\pi}\ln r_2$ (Nótese que están switcheados los signos)

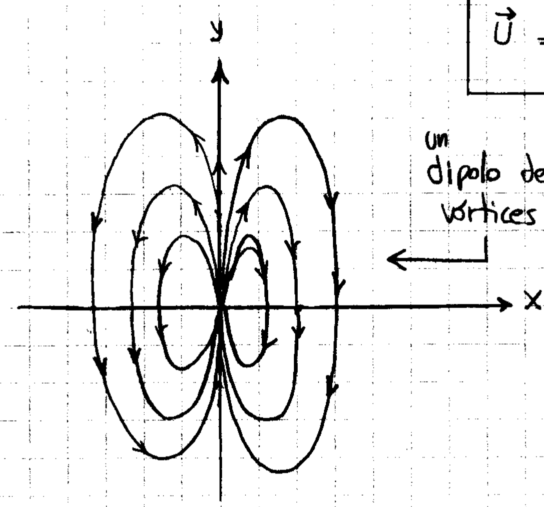
$\psi_T = \frac{-\Gamma}{2\pi}\left(\ln|\vec{r}+a\hat{x}| - \ln|\vec{r}-a\hat{x}|\right) \Rightarrow$ usando lo hecho para el dipolo de fuente/sumidero tendremos

$$\boxed{\underset{\text{first order}}{\vec{\Psi}_T} = -\frac{P.\cos\theta}{2\pi r}\,\hat{z}}$$ con $P = \Gamma 2a \Rightarrow$

luego $\vec{U} = \vec{\nabla}\times\Psi\hat{z} = \frac{1}{r}\cdot\frac{\partial\Psi}{\partial\theta}\,\hat{r} - \frac{\partial\Psi}{\partial r}\,\hat{\theta}$

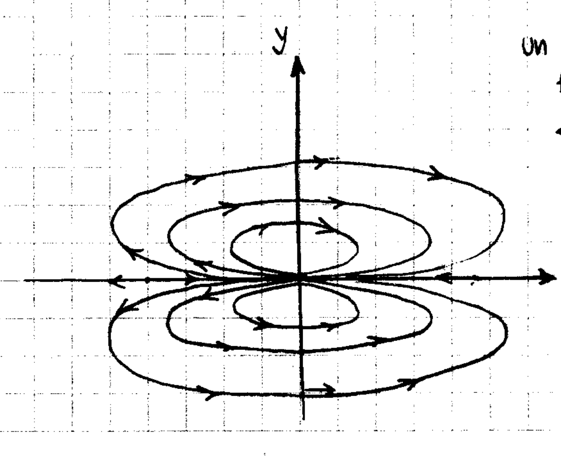
$$\vec{U} = \frac{1}{r}\cdot\frac{-P}{2\pi r}\cdot -\text{sen}\,\theta\,\hat{r} - \frac{(-)(-P)\cos\theta}{2\pi r^2}\,\hat{\theta}$$

$$\boxed{\vec{U} = \frac{P}{2\pi r^2}\left(+\text{sen}\,\theta\,\hat{r} - \cos\theta\,\hat{\theta}\right)}$$



un dipolo de vórtices $\longleftarrow$

$$\boxed{\text{rot}(\vec{u}) = \vec{\omega} = 0}$$



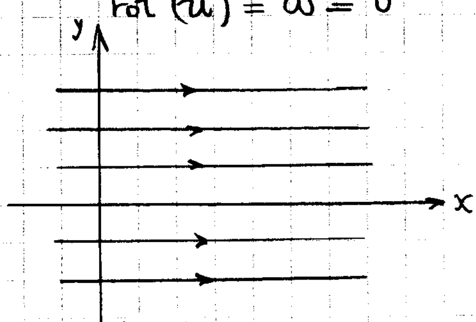
un dipolo de fuente/sumidero $\longleftarrow$

2. $\Psi(x,y) = \overset{①}{a\,y}\,,\ \overset{②}{b y^2}\,,\ \overset{③}{c x y}\,,\ \overset{④}{d(3x^2-y^2)y}$

① $\vec{u} = \mathrm{rot}(\Psi) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right)$

$\vec{u} = a\hat{x}$

$\mathrm{rot}(\vec{u}) = \vec{\omega} = 0$



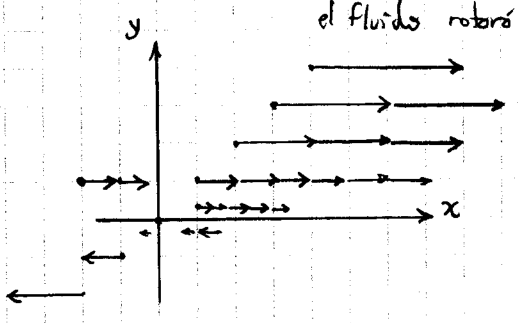
* Puntos de estancamiento serán $\begin{cases} u_x = 0 \\ u_y = 0 \end{cases}$

No tiene puntos de estancamiento pues $u_x \neq 0 \quad \forall (x,y)$

② $\vec{u} = 2by\,\hat{x}$

$\vec{\omega} = -\frac{\partial v_x}{\partial y} = -2b\,\hat{z}$

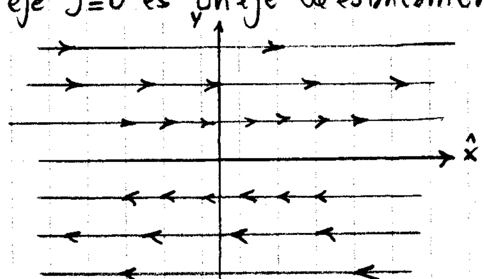
una partícula hechada en el fluido rotará CW



* Puntos de estancamiento serán

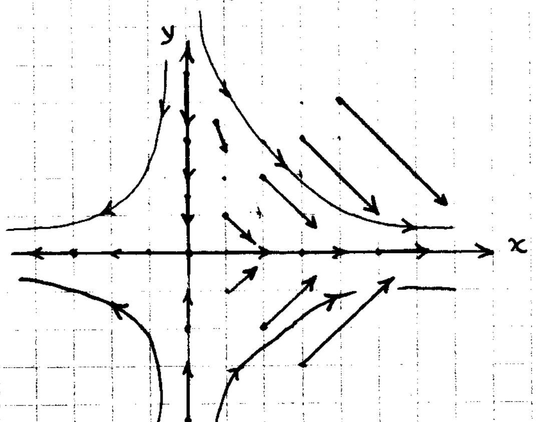
$y=0 \rightarrow \vec{u} = 2b.0\,\hat{x} = 0 \Rightarrow$

El eje $y=0$ es un eje de estancamiento



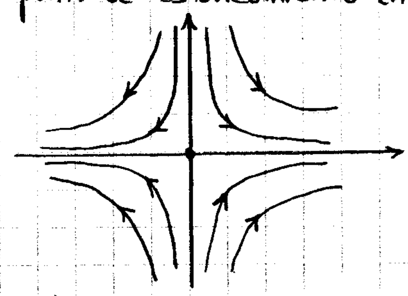
③ $\vec{u} = cx\,\hat{x} - cy\,\hat{y}$

$\mathrm{rot}(\vec{u}) = \vec{\omega} = 0$



* Puntos de estancamiento $\begin{cases} U_x = 0 \\ U_y = 0 \end{cases} \rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$

Hay punto de estancamiento en el origen



④ $\vec{u} = d\left([-2y]y + [3x^2-y^2]\right)\hat{x} - (dy\,3x)\hat{y}$

$= d(-2y^2+3x^2-y^2)\,\hat{x} - d3xy\,\hat{y}$

$\vec{u} = d(-3y^2+3x^2)\,\hat{x} - d3xy\,\hat{y}$

$\mathrm{rot}(\vec{u}) = \vec{\omega} = (-3dy + 3d2y)\,\hat{z} = 3dy\,\hat{z}$

3.

$$\vec{A} \equiv \Psi(x,y)\,\hat{z} \quad : \quad \mathrm{rot}(\Psi\hat{z}) = \vec{U}$$

$$\phi(x,y) \quad : \quad \mathrm{grad}(\phi) = \vec{U}$$

a) $$\mathrm{div}(\vec{U}) = \mathrm{div}\{\mathrm{rot}(\Psi\hat{z})\}$$

$$= \mathrm{div}\left(\frac{\partial\Psi}{\partial y}\hat{x} - \frac{\partial\Psi}{\partial x}\hat{y}\right) = \frac{\partial^2\Psi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y\partial x} = 0$$

b) $$\vec{U} = \mathrm{grad}\,\Psi \times \hat{z} = \left(\frac{\partial\Psi}{\partial x}\hat{x} + \frac{\partial\Psi}{\partial y}\hat{y}\right)\times\hat{z} = -\frac{\partial\Psi}{\partial x}\hat{y} + \frac{\partial\Psi}{\partial y}\hat{x}$$

$$\vec{U} = \frac{\partial\Psi}{\partial y}\hat{x} + -\frac{\partial\Psi}{\partial x}\hat{y}$$

$$\vec{U} = \mathrm{rot}(\Psi\hat{z})$$

c)

$$\mathrm{grad}(\phi) = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} \qquad \mathrm{rot}(\Psi\hat{z}) = \frac{\partial\Psi}{\partial y}\hat{x} + -\frac{\partial\Psi}{\partial x}\hat{y}$$

$$= U_x\hat{x} + U_y\hat{y} \qquad\qquad = U_x\hat{x} + U_y\hat{y}$$

$$\Rightarrow \boxed{\frac{\partial\phi}{\partial x} = \frac{\partial\Psi}{\partial y}} \wedge \boxed{\frac{\partial\phi}{\partial y} = -\frac{\partial\Psi}{\partial x}}$$ Condiciones Cauchy-Riemann

Son armónicas pues:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \qquad \nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} =$$

$$\nabla^2\phi = \partial_x U_x + \partial_y U_y \qquad \nabla^2\Psi = \partial_x(U_y) + \partial_y(U_x)$$

$$\nabla^2\phi = \partial_x\partial_y\Psi + \partial_y(\partial_x\Psi) = 0 \qquad \nabla^2\Psi = \frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0$$

$$-\partial^2_{xy}\phi + \partial^2_{yx}\phi = 0$$

$\Rightarrow \phi$ es armónica

$\Rightarrow \Psi$ es armónica

d) $W(z) = \phi(x,y) + i\,\Psi(x,y)$ con $\dfrac{dW}{dz} = \dfrac{\partial W}{\partial x} = -i\dfrac{\partial W}{\partial y}$

$z = x + iy$

$$\frac{dW}{dz} = \underbrace{\frac{\partial \phi}{\partial x} + i\frac{\partial \Psi}{\partial x}} = \frac{\partial \Psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$ (por estudios de variable $\mathbb{C}$)

pero $$\frac{\partial W}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \Psi}{\partial x} = \frac{dW}{dz}$$ y

$$-i\frac{\partial W}{\partial y} = -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \Psi}{\partial y}\right) = -i\frac{\partial \phi}{\partial y} + \frac{\partial \Psi}{\partial y} = \frac{dW}{dz}$$

e) $\tilde{U} = U_x + i\,U_y \Rightarrow$

$$\tilde{U} = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \frac{\partial \Psi}{\partial y} - i\frac{\partial \Psi}{\partial x}$$ luego conjugando

$$\tilde{U}^* = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x} + i\frac{\partial \Psi}{\partial x} = \frac{dW}{dz}$$

usamos condiciones de Cauchy-Riemann

4.

$$W(z) = \phi(x,y) + i\,\Psi(x,y)$$

Proviene de fluido incompresible

Proviene de irrotacionalidad $\vec{\omega} = 0$

$\boxed{W(z) = U_\infty^* . z}$

$\boxed{W(z) = +U_\infty . z . e^{-i\alpha}}$

a) Tiene rotor($\vec{u}$) nulo $\rightarrow \exists\ \phi(x,y) :\ \text{grad}(\phi) = \vec{u}$

$\therefore$ se tendrá que:
$$\begin{cases} \dfrac{\partial \phi}{\partial x} = U_\infty . \cos\alpha \\[2mm] \dfrac{\partial \phi}{\partial y} = U_\infty . \text{sen}\,\alpha \end{cases}$$

$U_\infty . \cos\alpha . z$
$- i\,U_\infty . \text{sen}\,\alpha . z$

$$\boxed{W(z) = U_\infty(\cos\alpha . x + \text{sen}\,\alpha . y) + i\,U_\infty(\cos\alpha . y - \text{sen}\,\alpha . x)}$$

b) Tiene $\text{rot}(\vec{u}) = 0 \rightarrow \exists\ \phi(x,y) :\ \text{grad}(\phi) = \vec{u}$

$\hat{r})\ \dfrac{\partial \phi}{\partial r} = \dfrac{Q}{2\pi r} \rightarrow \phi = \dfrac{Q}{2\pi} . \ln\left(\dfrac{r}{r_0}\right)$

$\hat{\theta})\ \dfrac{1}{r} . \dfrac{\partial \phi}{\partial \theta} = 0$

$\boxed{W(z) = \dfrac{Q}{2\pi} . \ln(z)}$

$$W(z) = \frac{Q}{2\pi} . \ln r + i . \frac{Q}{2\pi} . \theta \rightarrow \boxed{W(z) = \frac{Q}{2\pi}(\ln r + i\theta)}$$

con $\theta = \text{atan}(y/x)$
$r = \sqrt{x^2 + y^2}$

c) Tiene $\text{rot}(\vec{u}) = 0 \rightarrow \exists\ \phi(x,y) :\ \text{grad}(\phi) = \vec{u}$

$\hat{\theta})\ \dfrac{\Gamma}{2\pi r} = \dfrac{1}{r} . \dfrac{\partial \phi}{\partial \theta}$

$\Psi = -\dfrac{\Gamma}{2\pi} \ln r$

$\phi = \dfrac{\Gamma}{2\pi}\theta$

$\boxed{W(z) = \dfrac{\Gamma}{2\pi i} \ln(z)}$

$$\boxed{W(z) = \frac{\Gamma}{2\pi}(\theta - i\ln r)}$$, donde $\theta = \text{atan}(y/x)$
$r = \sqrt{x^2 + y^2}$

d) Ya habríamos evaluado $\phi, \Psi \Rightarrow$

$$W(z) = \phi + i\Psi \rightarrow$$

$$\boxed{W(r,\theta) = \frac{P.\cos\theta}{2\pi r} - i\,\frac{P.\operatorname{sen}\theta}{2\pi r}} \rightarrow W(z) = \frac{P}{2\pi}.e^{-i\theta}.\frac{1}{|z|} \rightarrow \boxed{W(z) = \frac{P}{2\pi}.\frac{1}{z}}$$

, donde $P = Q.d$ "momento" dipolar

e) faltaría evaluar ϕ : $\vec{\nabla}\phi = \vec{u} \rightarrow$

$$\frac{\partial\phi}{\partial r} = \frac{P}{2\pi r^2}.\operatorname{sen}\theta\ \hat{r}$$

$$\frac{1}{r}.\frac{\partial\phi}{\partial\theta} = \frac{-P}{2\pi r^2}.\cos\theta\ \hat{\theta}$$

$$\int\partial\phi = \frac{-P}{2\pi r}.\int\cos\theta.d\theta = \frac{-P}{2\pi r}.\operatorname{sen}\theta$$

$$\phi + C(r) = -\frac{P.\operatorname{sen}\theta}{2\pi r}$$

$$\frac{\partial\phi}{\partial r} = +\frac{P.\operatorname{sen}\theta}{2\pi r^2} - C'(r) = \frac{P}{2\pi r^2}.\operatorname{sen}\theta$$

$$C'(r) = 0$$

$$\boxed{\phi = -\frac{P.\operatorname{sen}\theta}{2\pi r}}$$

luego

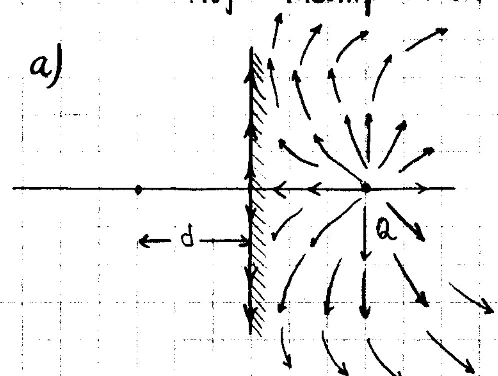
$$\boxed{W(r,\theta) = \frac{-P.\operatorname{sen}\theta}{2\pi r} - i.\frac{P.\cos\theta}{2\pi r}} \longrightarrow W(z) = \frac{P}{2\pi}.-i.\frac{1}{z} \rightarrow \boxed{W(z) = \frac{P}{2\pi}.\frac{1}{iz}}$$

5.

W, ϕ, Ψ, $\vec{u}$, puntos estancamiento, gráfico

flujo 2D
flujo incompresible $\Rightarrow \operatorname{div}(\vec{u}) = 0 \Rightarrow \exists\ \Psi\hat{z} : \operatorname{rot}(\Psi\hat{z}) = \vec{u}$

a)



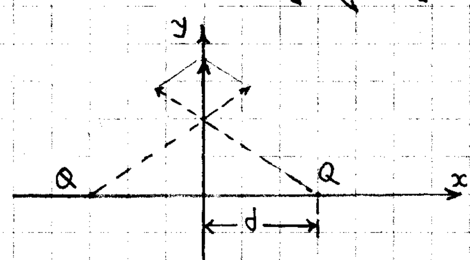
Fuente en el origen

$$W(z) = \frac{Q}{2\pi}.\ln(z) = \frac{Q}{2\pi}(\ln|z| + i\theta)$$

$$\boxed{W(z) = \frac{Q}{2\pi}\left(\ln(z-d) + \ln(z+d)\right)}$$

$$\boxed{\phi(x,y) = \frac{Q}{2\pi}\left(\ln\sqrt{(x-d)^2+y^2} + \ln\sqrt{(x+d)^2+y^2}\right)}$$

$$\boxed{\Psi(x,y) = \frac{Q}{2\pi}\left[\operatorname{atan}\left(\frac{y}{x-d}\right) + \operatorname{atan}\left(\frac{y}{x+d}\right)\right]}$$



$$\vec{u}^* = \frac{dW}{dz} = \frac{Q}{2\pi}\left(\frac{1}{z-d} + \frac{1}{z+d}\right)$$

$$\vec{u}^* = \frac{Q}{2\pi}\left(\frac{z^*-d}{|z-d|^2} + \frac{z^*+d}{|z+d|^2}\right)$$

Para que exista la línea de corriente necesito Q de igual valor

$$\boxed{\vec{u} = \frac{Q}{2\pi}\left(\frac{z-d}{|z-d|^2} + \frac{z+d}{|z+d|^2}\right)}$$

$$\frac{Q}{2\pi}\left(\frac{z-d}{|z-d|^2}+\frac{z+d}{|z+d|^2}\right)=0$$

$$\frac{z-d}{|z-d|^2} = -\frac{z+d}{|z+d|^2}$$

z : $\operatorname{Re} z > 0$

$$\frac{z-d}{z+d} = -\frac{|z-d|^2}{|z+d|^2}$$

$(x+iy)^2 = x^2 - y^2 + 2xyi$

$$\frac{(z-d)(z+d)}{\cancel{|z+d|^2}} = -\frac{|z-d|^2}{\cancel{|z+d|^2}}$$

$$z^2 - \cancel{zd} - d^2 + \cancel{zd} = -|z-d|^2$$

$$x^2 - y^2 + 2ixy - d^2 = -\ (x-d)^2 + y^2$$

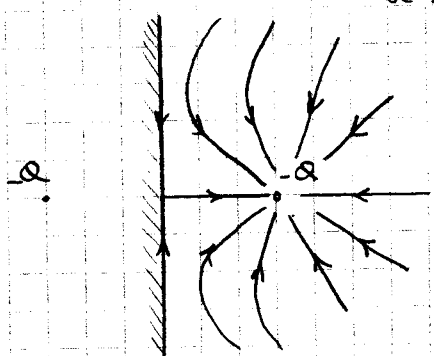
$$x^2 - \cancel{y^2} + 2ixy \cancel{-d^2} = -x^2 \cancel{-d^2} + 2xd \cancel{-y^2}$$

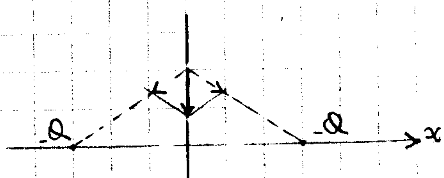
$x = d \longrightarrow$ Acá revienta

$2ixy = 0 \rightarrow \left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ el origen es punto de estancamiento

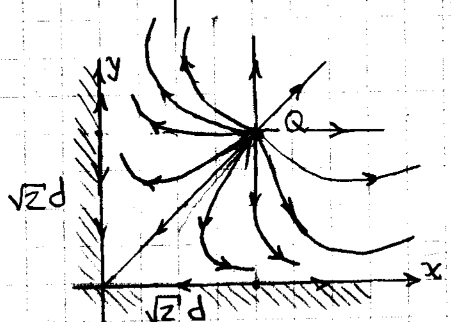
- Para el caso de un sumidero, se tiene una situación similar, tomando en cuenta el cambio de signo

$$W(z) = \frac{-Q}{2\pi}\left(\ln(z-d) + \ln(z+d)\right)$$





b)



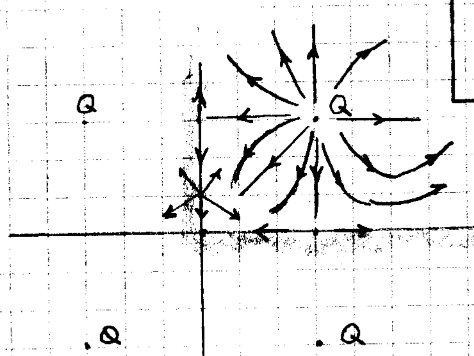
fuente en el origen

$$W(z) = \frac{Q}{2\pi}\ln(z) \rightarrow \text{con} \quad z_0 = \sqrt{2}d + i\sqrt{2}d \Rightarrow$$

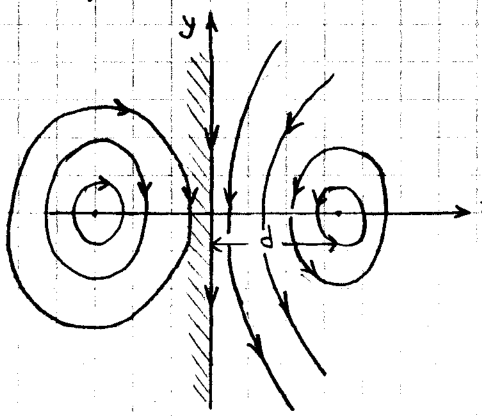
$$W(z) = \frac{Q}{2\pi}\left[\ln(z-z_0) + \ln(z-z_0^*) + \ln(z+z_0) + \ln(z+z_0^*)\right]$$

$$\boxed{W(z) = \frac{Q}{2\pi}\left(\ln\left(z-2de^{i\pi/4}\right) + \ln\left(z-2de^{i3\pi/4}\right) + \ln\left(z-2de^{i\frac{5}{4}\pi}\right) + \ln\left(z-2de^{i\frac{7}{4}\pi}\right)\right]}$$

$$= \frac{Q}{2\pi}\left(\ln\left[(x-\sqrt{2}d) + i(y-\sqrt{2}d)\right] + \right.$$



d)



Para un vórtice de circulación $\Gamma > 0$ tenemos:

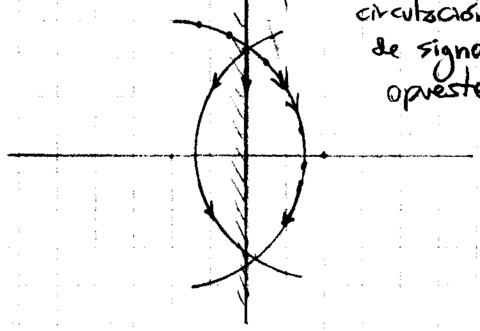
$\downarrow \Gamma < 0$

(en el origen) $W(z) = \frac{\Gamma}{2\pi i} \cdot \ln(z)$, $W = \frac{-\Gamma}{2\pi i} \cdot \ln(z)$

$$W(z) = \frac{\Gamma}{2\pi i}\left[\ln(z-d) - \ln(z+d)\right]$$

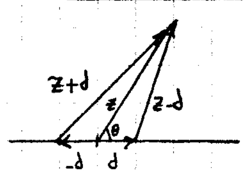
$$\boxed{W(z) = \frac{\Gamma}{2\pi i} \cdot \ln\left(\frac{z-d}{z+d}\right)} \rightarrow \text{potencial complejo}$$

Necesitamos una línea recta en el plano de separación $\Rightarrow$ necesito poner un vórtice de cada lado con circulación de signo opuesto



$$W(z) = -\frac{i\Gamma}{2\pi} \cdot \ln\left(\frac{z-d}{z+d}\right)$$

$$\boxed{\Psi(z) = -\frac{\Gamma}{2\pi} \cdot \ln\left|\frac{z-d}{z+d}\right|} \rightarrow \text{función corriente}$$



$|z-d| = |z|^2 + d^2 - 2|z|d.\cos\theta$

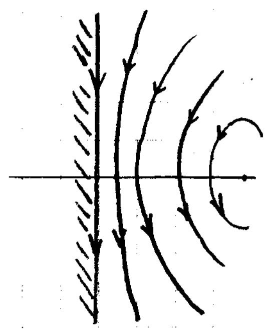
$|z+d| = |z|^2 + d^2 - 2|z|d.\cos(\pi-\theta)$

$|z+d| = |z|^2 + d^2 + 2|z|d.\cos\theta$

$\Rightarrow \left|\frac{z-d}{z+d}\right| = \text{Constante} \Leftrightarrow \theta = \frac{\pi}{2}$

$\therefore$ sobre la línea $\hat{y}$ es Ψ constante como corresponde

gráfico cualitativo líneas de corriente



$a+ib \cdot a-ib$

$$\frac{dW}{dz} = \frac{\Gamma}{2\pi i} \cdot \frac{1}{(z-d)} \cdot (z+d)$$

$$= -\frac{i\Gamma}{2\pi} \cdot \frac{(z+d)}{(z-d)} \cdot \frac{(z^*-d)}{(z^*-d)} = \frac{-i\Gamma\left(|z|^2 + dz^* - d^2 - dz\right)}{2\pi|z-d|^2}$$

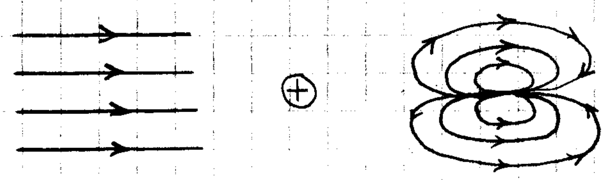
$$= \frac{-i\Gamma\left(-d(2iy) + x^2+y^2-d^2\right)}{2\pi\left[(x-d)^2+y^2\right]}$$

$$U^* = \frac{-\Gamma d 2y}{2\pi[(x-d)^2+y^2]} - \frac{i.(x^2+y^2-d^2)}{2\pi[(x-d)^2+y^2]}$$

$$\boxed{U = \frac{-\Gamma d 2y}{2\pi[(x-d)^2+y^2]} + \frac{i.(x^2+y^2-d^2)}{2\pi[(x-d)^2+y^2]}}$$

$\leftarrow$ campo de velocidades

6. Los contornos sólidos pueden simularse con un potencial tal que la frontera del mismo sea una línea de corriente.



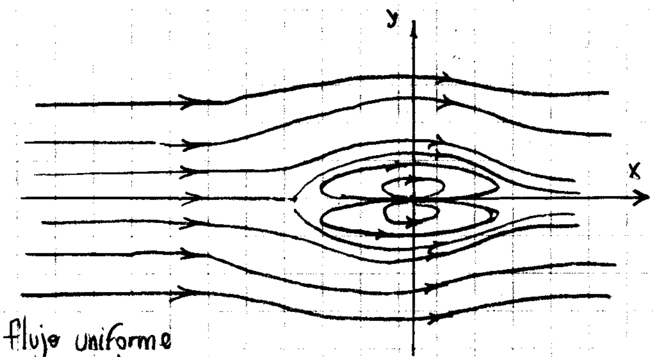
$$\phi_D = \mu \cdot \frac{\cos\theta}{r} \qquad \psi_D = -i\mu \cdot \frac{\operatorname{sen}\theta}{r}$$

$$\searrow W_D(z) = \mu \cdot (1/z)$$

$$\phi = U_\infty \cdot x \qquad U_\infty \in \mathbb{R}$$

$$\psi = U_\infty\, i\, y$$

$$\searrow W(z) = U_\infty \cdot z$$



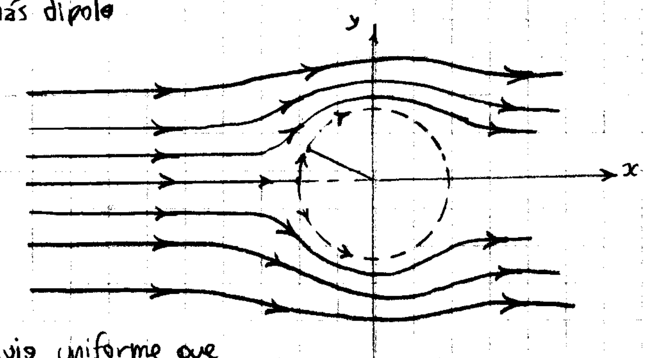
▲ flujo uniforme más dipolo

$$W_{TOTAL} = U_\infty z + \mu \cdot \frac{1}{z}$$

$$W_{TOTAL} = U_\infty \underset{x+iy}{z} + \mu \cdot \frac{\overset{x-iy}{z^*}}{|z|^2}$$

$$W_{Total} = \underbrace{\left(U_\infty + \frac{\mu}{|z|^2}\right)x}_{\phi_{total}} + \underbrace{i\left(U_\infty - \frac{\mu}{|z|^2}\right)y}_{\psi_{total}}$$

el potencial complejo total



▲ Flujo uniforme que embiste cilindro

Para el caso del cilindro embestido por el flujo uniforme, tendremos una línea de corriente sobre el contorno circular del mismo. Allí ψ constante.

Considerando el potencial del dipolo, veamos que

$$\frac{dW}{dz} = \tilde{U}^* = U_x - iU_y \quad , \text{ y pidamos } \quad \tilde{U} = 0 \rightarrow \tilde{U}^* = 0$$

$$= U_\infty + \mu \cdot \frac{d}{dz}\left(\frac{1}{z}\right) = U_\infty + \mu \cdot \frac{-1}{z^2} = 0 \Rightarrow$$

$$\frac{U_\infty}{\mu} = \frac{1}{z^2} \rightarrow z^2 = \frac{\mu}{U_\infty} = |z|^2 e^{i\theta 2} \Rightarrow$$

$$|z| = \sqrt{\frac{\mu}{U_\infty}} \qquad e^{i2\theta} = 1 = e^{i2n\pi} \Rightarrow \theta = n\pi$$

$$\theta = 0,\ \theta = \pi$$

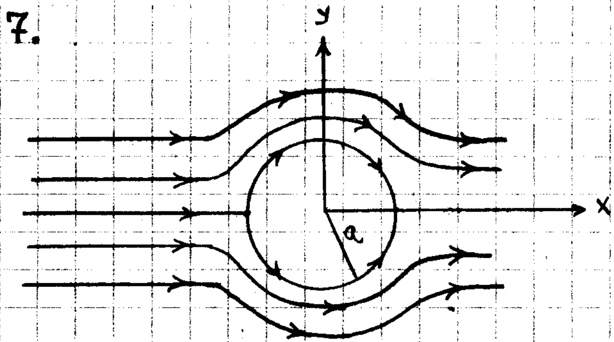
puntos de estancamiento $\rightsquigarrow z = \left(\frac{\mu}{U_\infty}\right)^{1/2} e^{i\pi} = -\left(\frac{\mu}{U_\infty}\right)^{1/2}$, y $z = +\left(\frac{\mu}{U_\infty}\right)^{1/2}$

Ahora considero la circunferencia $z = \left(\frac{\mu}{U_\infty}\right)^{1/2} \cdot e^{i\theta}$

Valdrá: $\psi\,(\text{circunferencia}) = \left(U_\infty - \frac{\cancel{\mu}}{\cancel{\mu}} \cdot U_\infty\right) \sqrt{\frac{\mu}{U_\infty}} \cdot \operatorname{sen}\theta = 0$

ψ es constante sobre la circunferencia de radio $a = \sqrt{\frac{\mu}{U_\infty}}$, y $\vec{U}$ tiene un punto de estancamiento en $-\sqrt{\frac{\mu}{U_\infty}}$ $\Rightarrow$ es equivalente al problema del cilindro de radio a embestido por un flujo uniforme.

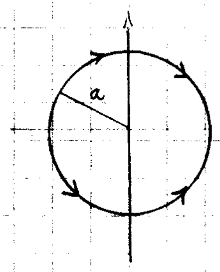
7.



Flujo que embiste a un cilindro.
El teorema del círculo nos dice que si tenemos un flujo sin singularidades en $|z| \leq a$ con $W_0 = f(z) \Rightarrow$ el potencial exterior será:

$$W(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

Ahora dentro de $|z| \leq a$ (el cilindro) no hay singularidades, el potencial complejo sin la "zona" $|z| \leq a$ es



$$W(z) = U_\infty^* . z = f(z)$$

$$\rightarrow \bar{f}(z) = U_\infty . z$$

$$\bar{f}(a^2/z) = U_\infty . \frac{a^2}{z} \quad \therefore$$

$$\boxed{W(z) = U_\infty^* z + U_\infty . \frac{a^2}{z}}$$

$$\frac{dW_z}{dz} = U_\infty^* + U_\infty . a^2 . -\frac{1}{z^2} = 0 \rightarrow$$

$$\frac{U_\infty a^2}{z^2} = U_\infty^* \qquad z^2 = \frac{U_\infty a^2}{U_\infty^*} = e^{i2\alpha} a^2$$

$$|z|^2 . e^{i2\theta} = a^2 . e^{i2\alpha} \quad , a \in \mathbb{R}$$

$$|z| = a \qquad \theta = \frac{2\alpha + 2n\pi}{2} = \alpha + n\pi \qquad n \text{ será } 0,1$$

Puntos de estancamiento

$$z = a . e^{i\alpha} \; , \; z = a . e^{i\alpha} . e^{i\pi}$$

$$\boxed{z = a . e^{i\alpha} \; ; \; z = -a . e^{i\alpha}}$$

$$\mathbb{F} = F_x - i F_y = \frac{i\rho}{2} \oint_{\mathcal{C}} \left|\frac{dW}{dz}\right|^2 dz \quad ; \quad F_x \hat{x} + F_y \hat{y} = \vec{F} = -\oint_{\mathcal{C}} p . \underbrace{\hat{n}\, dl}$$

$\hat{n}$ dl

Como es flujo estacionario e irrotacional vale Bernoulli,

$$P = -\frac{1}{2}\rho v^2 + K$$

pues no hay fuerzas $\Omega = 0$

$$\boxed{P + \rho\frac{v^2}{2} + \rho\Omega = K}$$

Aclaración

$$P = -\frac{\rho}{2}\left|U_\infty^* - \frac{U_\infty a^2}{z^2}\right|^2 + K \quad ; \text{ considero para simplificar } \alpha = 0 \Rightarrow$$

$$\left|U_\infty + \frac{U_\infty a^2}{|z|^2} . e^{-i2\theta}\right|^2 = \left|U_\infty + \frac{U_\infty a^2}{|z|^2} . \cos(2\theta) - i\frac{U_\infty a^2}{|z|^2} . \text{sen}(2\theta)\right|^2$$

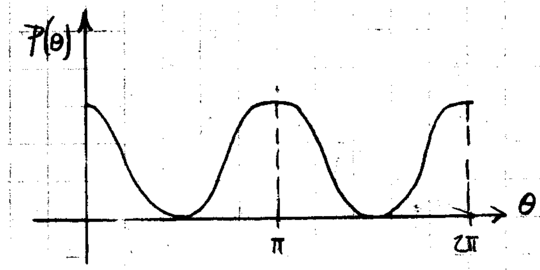
$$= (U_\infty)^2\left(1 + \frac{a^2}{|z|^2}\cos(2\theta)\right)^2 + U_\infty^2 \frac{a^4}{|z|^4} . \text{sen}^2(2\theta) =$$

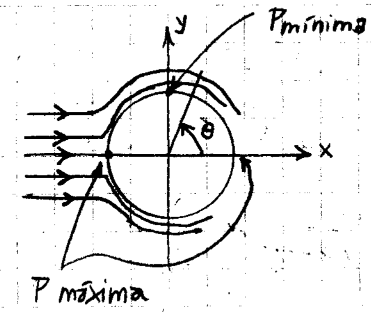
$$= U_\infty^2 + U_\infty^2 \frac{a^4}{|z|^4}\cos^2(2\theta) + 2U_\infty^2 \frac{a^2}{|z|^2}\cos(2\theta) + U_\infty^2 \frac{a^4}{|z|^4}\text{sen}^2(2\theta) = U_\infty^2\left(1 + \frac{a^4}{|z|^4} + \frac{2a^2}{|z|^2} . \cos 2\theta\right)$$

$$-\frac{\rho}{2}\left|\frac{dW}{dz}\right|^2 = -\frac{\rho}{2} . U_\infty^2\left(1 + \frac{a^4}{|z|^4} + \frac{2a^2}{|z|^2} . \cos(2\theta)\right) \quad , \text{ ahora hay que evaluar en } |z| = a$$

$$= \rho U_\infty^2 [1 + \cos(2\theta)] \rightarrow \boxed{p = \rho U_\infty^2 [1 + \cos(2\theta)] + K}$$

Esta será la distribución de presiones, que puede graficarse como:

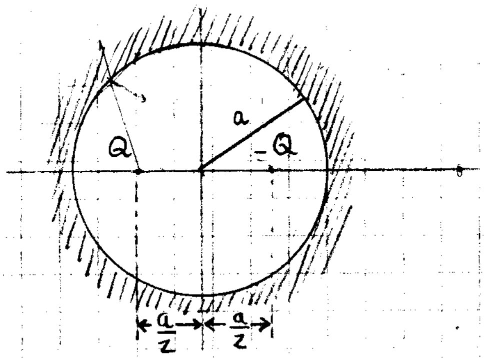




Todo esto parece bastante razonable

P_{max} en $\theta = 0, \pi$
P_{min} en $\theta = \pi/2$
$P(\theta = \pi/2) = P(\theta = \tfrac{3}{2}\pi)$

8.



Piden hallar ϕ

De entrada vemos que deberá haber línea de corriente en $|z| = a$

$$W_{f/s} = \frac{\pm Q}{2\pi} \ln(z) \rightarrow$$

$$W = \frac{Q}{2\pi}\left[-\ln\left(z - \frac{a}{2}\right) + \ln\left(z + \frac{a}{2}\right)\right]$$

$$W = \frac{Q}{2\pi}\left[\ln\left(\frac{z + a/2}{z - a/2}\right)\right]$$

Esto es para una fuente más un sumidero y como se ve necesitaremos "pegarle" algo más a fin de obtener $\psi = \text{cte.}$ en $|z| = a$ $\longleftarrow$

$$W = \underbrace{\frac{Q}{2\pi} \ln\left|\frac{z + a/2}{z - a/2}\right|}_{\phi} + \underbrace{i\,\theta . \frac{Q}{2\pi}}_{\psi} \qquad [1]$$

* **IDEA**

Supongamos que las f.y s. internas sean el potencial imagen obtenido mediante teorema del círculo de algunas f. y s. ubicadas fuera del cilindro.

Dada la simetría de las f.y s. internas ensayamos una suma de dos fuente y sumidero de q_1/q_2 desconocidas ubicadas sobre el eje real en c_1, c_2. Le pediremos que $|z| = a$ sea una línea de corriente.

$$W = \frac{q_1}{2\pi} . \ln(z - c_1) + \frac{q_2}{2\pi} . \ln(z + c_2) + \overbrace{\frac{q_1}{2\pi} \ln^*\left(\frac{a^2}{z} - c_1\right) + \frac{q_2}{2\pi} \ln^*\left(\frac{a^2}{z} + c_2\right)}^{\overline{f}\,(a^2/z)}$$

$$W(z = ae^{i\theta}) = \frac{q_1}{2\pi} \ln(ae^{i\theta} - c_1) + \frac{q_2}{2\pi} \ln(ae^{i\theta} + c_2) + \frac{q_1}{2\pi} \ln^*(ae^{-i\theta} - c_1) + \frac{q_2}{2\pi} \ln^*(ae^{-i\theta} + c_2)$$

$$\frac{q_1}{2\pi}\left(\ln[\#_1] + \ln^*[\#_1]\right) + \frac{q_2}{2\pi}\left(\ln[\#_2] + \ln^*[\#_2]\right)$$

$$\underbrace{\frac{q_1}{2\pi} 2 \ln|ae^{i\theta} - c_1| + \frac{q_2}{2\pi} 2 \ln|a.e^{i\theta} - c_2|}_{= \phi} \in \mathbb{R} \rightarrow \psi = 0 \rightarrow z = a.e^{i\theta} \text{ es una línea de corriente}$$

$$\frac{a^2}{z} - c_1 = \frac{a^2 - c_1.z}{z} \rightarrow \ln\left(\frac{a^2}{z} - c_1\right) = \ln(a^2 - c_1.z) - \ln(z) = \ln(c_1)\left(z - \frac{a^2}{c_1}\right) - \ln(z)$$

$$= \underset{\ln c_1 + i\pi}{\ln(-c_1)} + \ln\left(z - \frac{a^2}{c_1}\right) - \ln(z)$$

, en forma idem

$$\frac{a^2}{z} + c_2 \rightarrow = \ln(c_2) + \ln\left(z - \frac{a^2}{c_2}\right) - \ln(z),$$

Queremos que dentro del cilindro las imágenes sean nuestro problema original
$\Rightarrow$ pedimos:

[2] $\quad q_2 = Q \qquad q_1 = -Q \qquad C_2 = C_1 = 2a \;\rightarrow$

$$\overline{f}\left(\frac{a^2}{z^*}\right) = \frac{q_1}{2\pi}\left(\ln^*(-C_1) - \ln^*(z) + \ln^*\left(z - \frac{a^2}{C_1}\right)\right) + \frac{q_2}{2\pi}\left(\ln^*(C_2) - \ln^*(z^*) + \ln^*\left(z^* + \frac{a^2}{C_2}\right)\right)$$

$$-\frac{Q}{2\pi} i\pi - \frac{Q}{2\pi}\ln^*\left(z - \frac{a}{z}\right) + \frac{Q}{2\pi}\ln^*\left(z + \frac{a}{z}\right) = \frac{Q}{2\pi}\left[\ln^*\left(\frac{z+a/z}{z-a/z}\right) - i\pi\right]$$

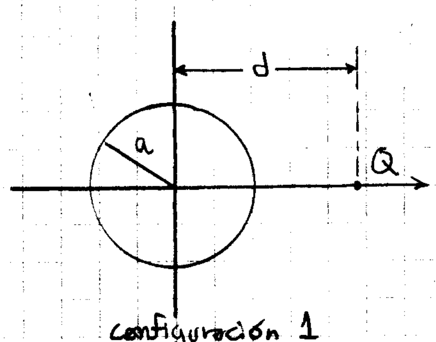
El $W(z)$ de las imágenes es, a menos de una constante, el de las cargas de nuestro problema original (con las elecciones hechas en [2]. En efecto:

sea $\ln(z) = \ln|z| + i\theta \;\rightarrow\; \ln^*(z) = \ln|z| - i\theta$

$$\Rightarrow W(z) = \frac{Q}{2\pi}\ln\left(\frac{z+2a}{z-2a}\right) + \frac{Q}{2\pi}\ln\left(\frac{z+a/z}{z-a/z}\right) - \frac{Q}{2\pi} i\pi$$

$$Re\left(W(z)\right) = \boxed{\frac{Q}{2\pi}\ln\left|\frac{z+2a}{z-2a}\right| + \ln\left|\frac{z+a/z}{z-a/z}\right| = \phi(z)}$$

9.



configuración 1

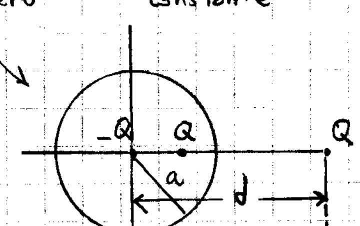
Este caso puede hacerse fácilmente usando teorema del círculo

1b)

$$W(z) = \frac{Q}{2\pi}\ln(z-d) + \frac{Q}{2\pi}\ln^*\left(\frac{a^2}{z} - d\right)$$

$$\ln^*\left(\frac{a^2 - dz}{z}\right) = \ln^*(a^2 - dz) - \ln^*(z)$$
$$= \ln^*\left(-d\left(\frac{a^2}{d} + z\right)\right) - \ln(z)$$
$$= \ln^*(-d) + \ln^*\left(z - \frac{a^2}{d}\right) - \ln^*(z)$$
$$= \ln(-d) + \ln\left(z - \frac{a^2}{d}\right) - \ln(z)$$

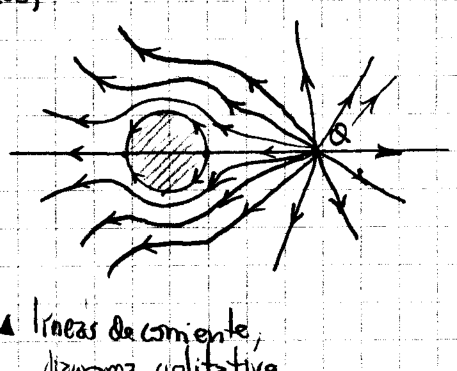
$$\boxed{W(z) = \frac{Q}{2\pi}\Big[\ln(z-d) + \underbrace{\ln\left(z - \frac{a^2}{d}\right) - \ln(z)}_{\text{fuente/sumidero imágenes}} + \underbrace{\ln(-d)}_{\text{una constante}}\Big]}$$



$\frac{a^2}{d} < a$

$a < d$ vale

1a)



▲ líneas de corriente, diagrama cualitativo

1c) $\quad \dfrac{dW}{dz} = \dfrac{Q}{2\pi}\left[\dfrac{1}{z-d} + \dfrac{1}{z - a^2/d} - \dfrac{1}{z}\right]$

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left[\frac{z^* - d}{|z-d|^2} + \frac{z^* - a^2/d}{|z - a^2/d|^2} - \frac{z^*}{|z|^2}\right]$$

pediremos $\dfrac{dW}{dz} = 0$ para los puntos de estancamiento

$$\frac{(z^*-d)\overbrace{|z|^2|z-a^2/d|^2}^{A}+(z^*-a^2/d)\overbrace{|z-d|^2|z|^2}^{B}-z^*\overbrace{|z-d|^2|z-a^2/d|^2}^{C}}{|z-d|^2\,|z-\frac{a^2}{d}|^2\,|z|^2}=0$$

$$z^*(A+B-C)=dA+\frac{a^2}{d}B$$

$$z^*=\frac{dA+(a^2/d)B}{A+B-C}$$

$$z^*=\frac{d|z|^2|z-a^2/d|^2+(a^2/d)|z-d|^2|z|^2}{|z|^2|z-a^2/d|^2+|z-d|^2|z|^2-|z-d|^2|z-a^2/d|^2}$$

Pero $z^*=z \rightarrow z\in\mathbb{R} \Rightarrow z=x$

$$x=\frac{dx^2(x-a^2/d)^2+(a^2/d)(x-d)^2x^2}{x^2(x-\frac{a^2}{d})^2+(x-d)^2x^2-(x-d)^2(x-\frac{a^2}{d})^2}$$

$$x=\frac{d(x-a^2/d)^2+(a^2/d)(x-d)^2}{(x-\frac{a^2}{d})^2+(x-d)^2-\underbrace{\frac{(x-d)^2}{x^2}(x-a^2/d)^2}}$$

$\rightarrow (x^2-2xd+d^2)(x^2-2\frac{a^2}{d}x+\frac{a^4}{d^2})$

$x^4-2x^3d+d^2x^2$

$-2\frac{a^2}{d}x^3+4a^2x^2-2xa^2d$

$x^2\frac{a^4}{d^2}-2x\frac{a^4}{d}+a^4$

Esta cuenta es por amor al cálculo

denom. = $\cancel{x^2}-\cancel{2x\frac{a^2}{d}}+\frac{a^4}{d^2}+x^2+\cancel{d^2}-\cancel{2xd}$

$-\cancel{x^2}+\cancel{2xd}-\cancel{d^2}+\cancel{2\frac{a^2}{d}x}-4a^2+2\frac{a^2d}{x}$

$-\cancel{\frac{a^4}{d^2}}+\frac{2a^4}{xd}-\frac{a^4}{x^2}=x^2-4a^2+2\frac{a^2d}{x}+\frac{2a^4}{xd}-\frac{a^4}{x^2}$

numer. = $dx^2-2xa^2+\frac{a^4}{d}+\frac{a^2}{d}x^2+a^2d-a^22x$

$=x^2(d+\frac{a^2}{d})+\frac{a^4}{d}+a^2d-4xa^2$

$x^3-\cancel{4a^2x}+2a^2d+2\frac{a^4}{d}-\frac{a^4}{x}=x^2d-\cancel{4xa^2}+\cancel{\frac{a^4}{d}}+\frac{a^2}{d}x^2+\cancel{a^2d}$

$a^2d+\frac{a^4}{d}+x^3-\frac{a^4}{x}=x^2d-x^2\frac{a^2}{d}$

$xa^2d+x\frac{a^4}{d}+x^4-a^4=x^3d-x^3\frac{a^2}{d}$

$x^4+x(a^2d+\frac{a^4}{d})+x^2(\frac{a^2}{d}-d)-a^4=0$

$x^4-a^4+(xa^2-x^3)(d+\frac{a^2}{d})=0$

$(x^4-a^4)-x[x^2-a^2][d+\frac{a^2}{d}]=0$

$[x^2-a^2][x^2+a^2]-x[x^2-a^2][d+\frac{a^2}{d}]=0$

$[x^2-a^2]\left(x^2+a^2-x[d+\frac{a^2}{d}]\right)=0$

$\boxed{x=\begin{matrix}a\\-a\end{matrix}}$

$$x=\frac{d+\frac{a^2}{d}\pm\sqrt{(d+\frac{a^2}{d})^2-4a^2}}{2}=\frac{(d+\frac{a^2}{d})\pm|d-\frac{a^2}{d}|}{2}$$

$d^2+2a^2+\frac{a^4}{d^2}$

$\frac{d+\frac{a^2}{d}+d-\frac{a^2}{d}}{2}=$ $\boxed{x=d}$

$\boxed{x=\frac{a^2}{d}}$

Pero obviamente los únicos con significado físico para este problema serán

$$\boxed{z=a \quad , \quad z=-a}$$

1d)

$$p=-\rho\frac{v^2}{2}+K=-\frac{\rho}{2}\left|\frac{dW}{dz}\right|^2+K \;;\quad \text{pero evaluado en } z=a\,e^{i\theta}$$

$$\left.\frac{dW}{dz}\right|_{z=ae^{i\theta}}=\frac{Q}{2\pi}\left[\frac{ae^{-i\theta}-d}{\underbrace{|ae^{i\theta}-d|^2}_{a^2\left|e^{i\theta}-\frac{d}{a}\right|^2}}+\frac{ae^{-i\theta}-a^2/d}{\underbrace{\left|ae^{i\theta}-\frac{a^2}{d}\right|^2}_{a^2\left|e^{i\theta}-\frac{a}{d}\right|^2}}-\frac{\cancel{a}e^{-i\theta}}{a^{\cancel{2}}}\right]$$

$$(\cos\theta-\#)^2+\operatorname{sen}^2\theta$$
$$1-2\#\cos\theta+\#^2$$

$$\left(\frac{e^{-i\theta}-d/a}{\underbrace{a\left(1-2\frac{d}{a}\cos\theta+\frac{d^2}{a^2}\right)}_{\cancel{a}\cdot\frac{d}{\cancel{a}}\left(\frac{a}{d}-2\cos\theta+\frac{d}{a}\right)}}+\frac{e^{-i\theta}-a/d}{\underbrace{a\left(1-2\frac{a}{d}\cos\theta+\frac{a^2}{d^2}\right)}_{a\cdot\frac{a}{d}\left(\frac{d}{a}-2\cos\theta+\frac{a}{d}\right)}}-\frac{e^{-i\theta}}{a}\right)$$

$$\frac{\cancel{\frac{a}{d}e^{-i\theta}}-\frac{1}{\cancel{a}}\cancel{a}\quad+\cancel{\frac{d}{a}e^{-i\theta}}\quad-\frac{\cancel{a}}{\cancel{d}}\cdot\frac{\cancel{d}}{\cancel{a}}\quad-\cancel{e^{-i\theta}\frac{a}{d}}+2\cos\theta\,e^{-i\theta}-\cancel{\frac{d}{a}e^{-i\theta}}}{\left(\frac{a}{d}-2\cos\theta+\frac{d}{a}\right)a}$$

$$\frac{-2\left(1-\cos\theta.e^{-i\theta}\right)}{}=\frac{-2\left(1-\cos^2\theta+i.\cos\theta.\operatorname{sen}\theta\right)}{a\left(\frac{a}{d}-2\cos\theta+\frac{d}{a}\right)}=\frac{\left(\overbrace{1-2\cos^2\theta+\cos^4\theta}^{-\cos(2\theta)}+\overbrace{\cos^2\theta.\operatorname{sen}^2\theta}^{\frac{\operatorname{sen}^2(2\theta)}{4}}\right)}{\left|\frac{a}{2}\right|\Big(\qquad\Big)^2}$$

$$\left|\left.\frac{dW}{dz}\right|_{z=ae^{i\theta}}\right|^2=\frac{}{\frac{a^2}{4}\left(\frac{a}{d}-2\cos\theta+\frac{d}{a}\right)^2}$$

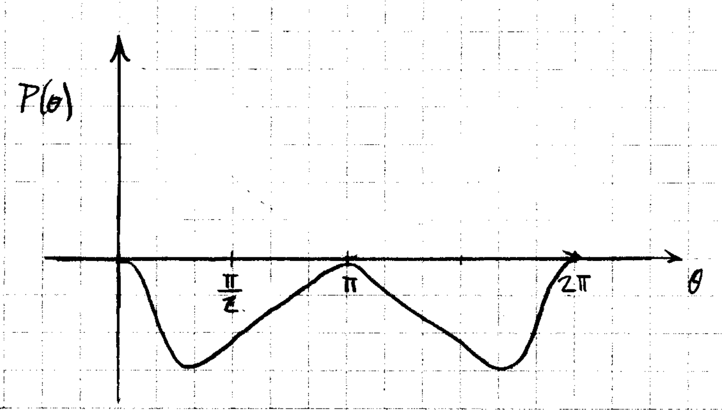
$$=\left|\frac{2/a}{\left(\frac{a}{d}-2.\cos\theta+\frac{d}{a}\right)}.\left(-\operatorname{sen}\theta-i.\operatorname{sen}\theta.\cos\theta\right)\right|^2$$

$$\left(\frac{+2/a.\operatorname{sen}\theta}{\left|\frac{a}{d}-2.\cos\theta+\frac{d}{a}\right|}\right)^2\underbrace{\left|\left[1+i.\cos\theta\right]\right|^2}_{(1+\cos^2\theta)}$$

$$\frac{(4/a^2).\operatorname{sen}^2\theta.(1+\cos^2\theta)}{\left(\frac{a}{d}+\frac{d}{a}-2.\cos\theta\right)^2}$$

Presión sobre el contorno sólido

$$P(\theta)=-\frac{\rho}{2}\cdot\frac{4}{a^2}\cdot\frac{\operatorname{sen}^2\theta.(1+\cos^2\theta)}{\left(\frac{a}{d}+\frac{d}{a}-2.\cos\theta\right)^2}+K$$

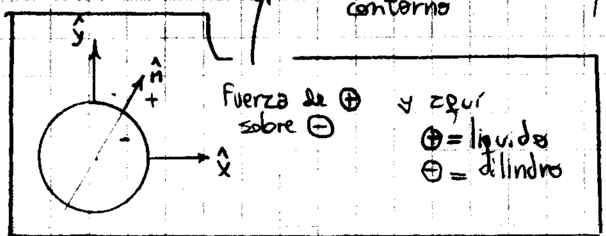


Podríamos hacer $P(\theta)$ positiva sumándole la constante K

1e)

$$\mathbb{F} = F_x - iF_y = \frac{i\rho}{2} \oint_\Gamma \left(\frac{dW}{dz}\right)^2 dz$$

o bien $$\vec{F} = -\int_{\text{contorno}} p \cdot \hat{n}\, d\ell = -\int_\Gamma p(\theta)\, r\, d\theta \left(\overbrace{\cos\theta\,\hat{x} + \operatorname{sen}\theta\,\hat{y}}^{\hat{r}}\right)$$



en $\Gamma = a\,e^{i\theta}$

Usando la distribución de presión en el borde del cilindro ($|z| = a$) será:

$$F_x = \int_0^{2\pi} \frac{\rho\, 2}{a^2} \frac{\left(\operatorname{sen}^2\theta + \operatorname{sen}^2\theta \cdot \cos\theta\right)}{\left(\frac{a}{d} + \frac{d}{a} - 2\cos\theta\right)} \cdot a \operatorname{sen}\theta \, d\theta$$

$$F_x = \frac{2\rho}{a^2}\left[\int_0^{2\pi} \frac{\operatorname{sen}^3\theta \cdot a\, d\theta}{\left(\frac{a}{d} + \frac{d}{a} - 2\cos\theta\right)} + \int_0^{2\pi} \frac{\operatorname{sen}^3\theta \cdot a \cos\theta}{\left(\frac{a}{d} + \frac{d}{a} - 2\cos\theta\right)} d\theta\right]$$

Estas integrales son básicamente un asco, probaremos con la teoría de variable compleja:

$$\oint_\Gamma \left(\frac{dW}{dz}\right)^2 dz = 2\pi i \cdot \operatorname{Res}\left\{\left(\frac{dW}{dz}\right)^2, (z = 0,\ z = a^2/d)\right\}$$

$$\left(\frac{dW}{dz}\right)^2 = \left(\frac{Q}{2\pi}\right)^2 \Bigg(\underbrace{\frac{-1}{(z-d)}\cdot\frac{1}{(z)}} + \frac{1}{(z-d)}\cdot\frac{1}{(z-a^2/d)} + \underbrace{\frac{1}{(z-d)^2}}$$

$$- \underbrace{\frac{1}{(z-a^2/d)(z)}} + \frac{1}{(z-a^2/d)(z-d)} + \underbrace{\frac{1}{(z-a^2/d)^2}}$$

$$\underbrace{\frac{-1}{(z)(z-d)}} + \underbrace{\frac{-1}{(z-a^2/d)(z)}} + \underbrace{\frac{1}{z^2}} \Bigg)$$

$$\left(\frac{dW}{dz}\right)^2 = \left(\frac{Q}{2\pi}\right)^2 \left[\underset{\text{Polo orden 1}}{\frac{-2/(z-d)}{z}} - \underset{\substack{\text{polo orden 1 en 0} \\ \text{" " " " } a^2/d}}{\frac{2}{(z)(z-a^2/d)}} + \underset{\text{polo orden 1 en } a^2/d}{\frac{2/(z-d)}{(z-a^2/d)}} + \underset{\substack{\text{polo de} \\ \text{orden 2} \\ \text{en 0}}}{\frac{1}{z^2}} + \underset{\substack{\text{polo de orden} \\ 2 \text{ en } a^2/d}}{\frac{1}{(z-a^2/d)^2}} \right.$$

$$\left. \frac{1}{(z-d)^2} \right]$$

$$\operatorname{Res}(f, z=0) + \operatorname{Res}(f, z=a^2/d) =$$

$$\lim_{z\to 0}\left[\left(\frac{Q}{2\pi}\right)^2 \left(\frac{-2/(z-d)\cdot \cancel{z}}{\cancel{z}} - \frac{2/(z-a^2/d)\cdot\cancel{z}}{\cancel{z}}\right)\right] + \Bigg(\underbrace{\frac{1}{1!}\cdot\frac{d}{dz}\left[\left(\frac{1}{\cancel{z^2}}\right)\cancel{z^2}\right]}_{=0}\Bigg)\left(\frac{Q}{2\pi}\right)^2 +$$

$$= \frac{Q^2}{(2\pi)^2}\left(\frac{2}{d} + \frac{2d}{a^2}\right)$$

$$\lim_{z\to a^2/d}\left[\left(\frac{Q}{2\pi}\right)^2 \left(-\frac{2/z}{\cancel{(z-a^2/d)}}\cancel{(z-a^2/d)} + \frac{2/z-d}{\cancel{(z-a^2/d)}}\cancel{(z-a^2/d)}\right)\right] + \Bigg(\underbrace{\frac{1}{1!}\frac{d}{dz}\left(\frac{1}{\cancel{(z-a^2/d)^2}}\cancel{(z-a^2/d)^2}\right)}_{=0}\Bigg)\left(\frac{Q}{2\pi}\right)^2$$

$$= \frac{Q^2}{(2\pi)^2}\left(-\frac{2d}{a^2} + \frac{2}{\frac{a^2}{d} - d}\right)$$

Sumando todos los residuos tendremos

$$\mathrm{Res}\left\{\left(\frac{dW}{dz}\right)^2, (z=0, z=a^2/d)\right\} = \frac{Q^2}{4\pi^2}\left(\frac{2}{d} + \cancel{\frac{2d}{a^2}} - \cancel{\frac{2d}{a^2}} + \frac{2d}{a^2-d^2}\right)$$

$$\frac{2a^2 - \cancel{2d^2} + \cancel{2d^2}}{d(a^2-d^2)} = \frac{2a^2}{d(a^2-d^2)}$$

$$= \frac{Q^2 a^2}{2\pi^2 (a^2-d^2)\, d}$$

$$\mathbb{F} = \frac{i\rho}{\cancel{2}} \cdot \frac{Q^2 a^2 \cdot \cancel{2\pi} i}{2\pi^2 d\,(a^2-d^2)} \rightarrow \mathbb{F} = F_x - iF_y \Rightarrow$$

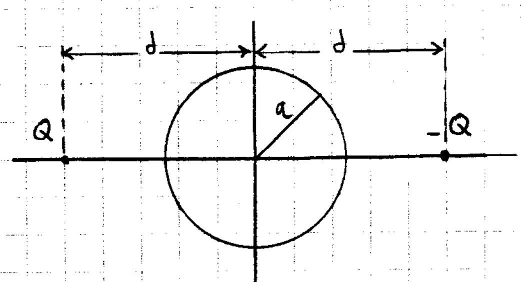
$$= -\frac{\rho}{2\pi}\frac{Q^2 a^2}{d\,(a^2-d^2)}$$

$$\boxed{F_x = \frac{\rho}{2}\frac{Q^2 a^2}{\pi d\,(d^2-a^2)}}$$

$$F_y = 0$$

La fuerza es en $+\hat{x}$

2 b)



Nuevamente podemos usar teorema del círculo

$$W(z) = \frac{Q}{2\pi}\ln(z+d) - \frac{Q}{2\pi}\ln(z-d) + \frac{Q}{2\pi}\ln^*\left(\frac{a^2}{z}+d\right) - \frac{Q}{2\pi}\ln^*\left(\frac{a^2}{z}-d\right)$$

Ahora serán.

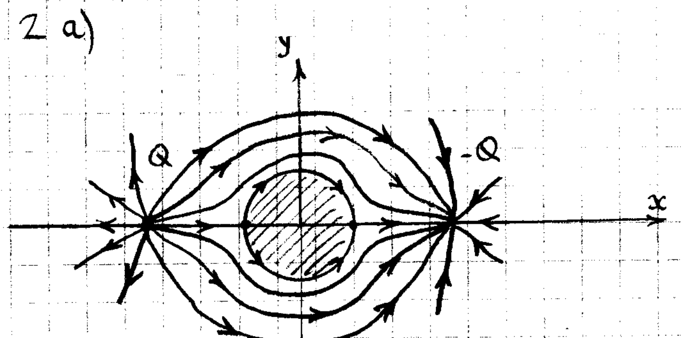
$$\ln^*\left(\frac{a^2}{z}+d\right) = \ln^*(a^2+dz) - \ln^*(z) = -\ln^*(z) + \ln^*\left(z+\frac{a^2}{d}\right) + \ln^*(d)$$

$$\ln^*\left(\frac{a^2}{z}-d\right) = -\ln^*(z) + \ln^*\left(z-\frac{a^2}{d}\right) + \ln^*(-d)$$

$$\boxed{W(z) = \frac{Q}{2\pi}\left[\ln\left(\frac{z+d}{z-d}\right) + \ln\left(\frac{z+a^2/d}{z-a^2/d}\right) + \ln\left(\frac{d}{-d}\right)\right]}$$

una constante

2 a)



2 c)

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left[\cancel{\frac{z-d}{z+d}}\left(\frac{1(z-d)-(z+d)1}{(z-d)^{\cancel{2}}}\right) + \cancel{\frac{z-a^2/d}{z+a^2/d}}\left(\frac{1(z-a^2/d)-(z+a^2/d)1}{(z-a^2/d)^{\cancel{2}}}\right)\right]$$

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left[\frac{-2d}{z^2-d^2} + \frac{-2a^2/d}{(z^2-a^4/d^2)}\right]$$

Sería de esperar que haya puntos de estancamiento en $z = a, -a$; pero hagamos la cuenta para convencernos :

$$\frac{Q}{2\pi}\left(-\frac{2d}{z^2-d^2} - \frac{2a^2/d}{z^2-\frac{a^4}{d^2}}\right) = 0 \rightarrow \frac{\cancel{2}d}{z^2-d^2} = \frac{-\cancel{2}a^2}{d\left(z^2-\frac{a^4}{d^2}\right)}$$

$$z^2 d^2 - a^4 = -z^2 a^2 + a^2 d^2$$
$$z^2 (d^2 + a^2) = +a^2 d^2 + a^4$$
$$z^2 = \frac{a^2 \cancel{(d^2+a^2)}}{\cancel{a^2+d^2}} \rightarrow z^2 = a^2$$
$$z = a . e^{ik\pi} \qquad k = 0,1 \rightarrow$$

Puntos de estancamiento $\boxed{z = \pm a}$

2d)

$$P = -\frac{\rho}{2} \left| \frac{dW}{dz} \right|^2 + K \qquad \text{evaluando en } z = a.e^{i\theta}$$

$$P(\theta) = -\frac{\rho}{2} \left| \frac{Q}{\cancel{2}\pi} \left(\frac{-\cancel{2}d}{a^2 e^{i2\theta} - d^2} + \frac{-\cancel{2}a^2/d}{a^2 e^{i2\theta} - \frac{a^4}{d^2}} \right) \right|^2$$

$$-\frac{\rho}{2} \cdot \frac{Q^2}{\pi^2} \left| \frac{-d}{a^2 \left(e^{i2\theta} - \frac{d^2}{a^2}\right)} - \frac{a^2/d}{a^2 \left(e^{i2\theta} - \frac{a^2}{d^2}\right)} \right|^2$$

$$P(\theta) = -\frac{\rho}{2} \frac{Q^2}{\pi^2} \left(\frac{1}{a^2}\right)^2 \underbrace{\left| -\frac{d}{\left(e^{i2\theta} - \frac{d^2}{a^2}\right)} - \frac{a^2}{d\left(e^{i2\theta} - \frac{a^2}{d^2}\right)} \right|^2}_{\equiv u}$$

$$u \equiv -\frac{d\left(e^{-i2\theta} - d^2/a^2\right)}{\left|e^{i2\theta} - \frac{d^2}{a^2}\right|^2} - \frac{a^2\left(e^{-i2\theta} - a^2/d^2\right)}{d\left|e^{i2\theta} - \frac{a^2}{d^2}\right|^2} = -\left(\frac{d\left(\quad\right)}{\frac{d^2}{a^2}\left(\frac{a^2}{d^2} - 2\cos(2\theta) + \frac{d^2}{a^2}\right)} + \frac{\frac{a^2}{d}\left(\quad\right)}{\frac{a^2}{d^2}\left(\frac{d^2}{a^2} - 2\cos(2\theta) + \frac{a^2}{d^2}\right)} \right)$$

$$\left(\cos(2\theta) - \frac{d^2}{a^2}\right)^2 + \operatorname{sen}^2(2\theta)$$
$$1 - 2\frac{d^2}{a^2}\cos(2\theta) + \frac{d^4}{a^4}$$

$$= -\frac{\frac{\cancel{d}a^2}{d^{\cancel{2}}}\left(e^{-i2\theta} - d^2/a^2\right) + \frac{\cancel{a^2}d^{\cancel{2}}}{\cancel{d}\,\cancel{a^2}}\left(e^{-i2\theta} - a^2/d^2\right)}{\left(\frac{a^2}{d^2} - 2\cos(2\theta) + \frac{d^2}{a^2}\right)}$$

$$= \frac{\frac{a^2}{d} e^{-i2\theta} - d + d.e^{-i2\theta} - \frac{a^2}{d}}{}$$

$$= \frac{\left(e^{-i2\theta} - 1\right)\left(\frac{a^2}{d} - d\right)}{\left(\frac{a^2}{d^2} - 2\cos(2\theta) + \frac{d^2}{a^2}\right)} \Rightarrow$$

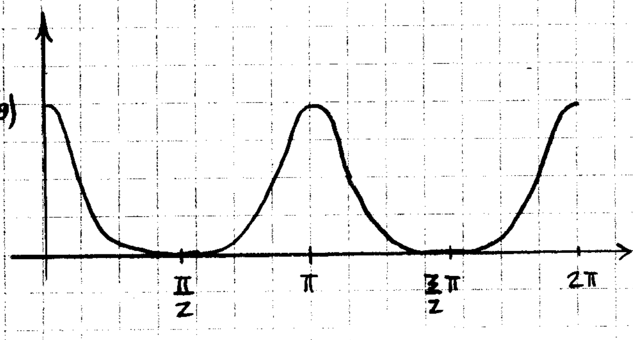
$$\cos^2(2\theta) + \operatorname{sen}^2(2\theta) + 2\cos(2\theta) + 1$$

$$P(\theta) = -\frac{\rho}{2} \frac{Q^2}{\pi^2 a^4} \frac{(a^2 - d^2)^2}{d^2 \left(\frac{a^2}{d^2} - 2\cos(2\theta) + \frac{d^2}{a^2}\right)^2} \cdot \left(\left(\cos(2\theta) + 1\right)^2 + \operatorname{sen}^2(2\theta)\right)$$

$$P(\theta) = -\frac{\rho}{\cancel{2}} \cdot \frac{Q^2 (a^2 - d^2)^2}{\pi^2 a^4 d^2} \frac{\cancel{2}\left[1 + \cos(2\theta)\right]}{\left(\frac{a^2}{d^2} - 2\cos(2\theta) + \frac{d^2}{a^2}\right)^2}$$

$$P(\theta) = -\frac{\rho Q^2}{\pi^2} \cdot \frac{(a^2 - d^2)^2}{a^4 d^2} \frac{2 \cos^2\theta}{\left(\frac{a^2}{d^2} + \frac{d^2}{a^2} - 2\cos(2\theta)\right)^2}$$

Podemos graficar esta función



2e)

$$\mathbb{F} = \frac{i\rho}{2}\oint_{\Gamma = a.e^{i\theta}} \left(\frac{dW}{dz}\right)^2 . dz = \overbrace{\frac{i\rho}{\cancel{2}} . \cancel{2}\pi i}^{-\rho.\pi} . \operatorname{Res}\left\{ \left(\frac{dW}{dz}\right)^2 , \left(z = \frac{a^2}{d} , z = -\frac{a^2}{d} \right) \right\}$$

$$\left(\frac{dW}{dz}\right)^2 = +\frac{Q^2}{\pi^2}\left[\frac{d}{(z^2-d^2)} + \frac{a^2/d}{(z^2-a^4/d^2)} \right]\left[\frac{d}{(z^2-d^2)} + \frac{a^2/d}{(z^2-a^4/d^2)} \right]$$

$$\left(\frac{dW}{dz}\right)^2 = +\frac{Q^2}{\pi^2}\left(\underbrace{\frac{d^2}{(z^2-d^2)^2}}_{\text{No tiene sing. en int}(\Gamma)} + \frac{a^2 . 2}{(z^2-d^2)(z^2-a^4/d^2)} + \frac{a^4/d^2}{(z^2-a^4/d^2)^2} \right)$$

$$\frac{2a^2/(z^2-d^2)}{\left(z-\frac{a^2}{d}\right)\left(z+\frac{a^2}{d}\right)} + \frac{a^4/d^2}{\left(z-\frac{a^2}{d}\right)^2\left(z+\frac{a^2}{d}\right)^2}$$

Polo simple en $\begin{cases} a^2/d \\ -a^2/d \end{cases}$ polo de orden 2 en a^2/d

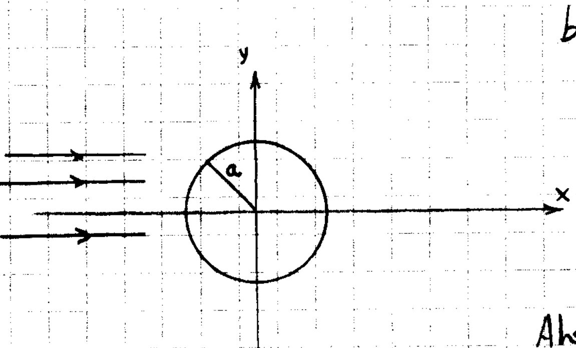
" " " " " $-a^2/d$

$$\left.\begin{aligned} &\lim_{z \to a^2/d} \frac{Q^2}{\pi^2}\left(\frac{2a^2}{(z^2-d^2)\left(z+\frac{a^2}{z}\right)} \right) = \frac{Q^2 \; 2\cancel{a^2}}{\pi^2\left(\frac{a^4}{d^2}-d^2\right)\left(\frac{\cancel{a^2}2}{\cancel{d}}\right)} \\ &\lim_{z \to -a^2/d} \frac{Q^2}{\pi^2}\left(\frac{2a^2}{(z^2-d^2)\left(z-a^2/z\right)} \right) = \frac{Q^2 \; 2\cancel{a^2}}{-\pi^2\left(\frac{a^4}{d^2}-d^2\right)\frac{\cancel{a^2}2}{\cancel{d}}} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{suma} = 0 \end{aligned}\right\} \text{Aporte al Res de los polos de orden 1}$$

$$\left.\begin{aligned} &\frac{d}{dz}\left(\frac{a^4/d^2}{\left(z+\frac{a^2}{d}\right)^2} \right)\Bigg|_{z=\frac{a^2}{d}} + \frac{d}{dz}\left(\frac{a^4/d^2}{\left(z-\frac{a^2}{d}\right)^2} \right)\Bigg|_{z=-\frac{a^2}{z}} = \frac{-2a^4/d^2}{\left(2\frac{a^2}{d}\right)^3} - \frac{2a^4/d^2}{\left(-2\frac{a^2}{d}\right)^3} \\ &= 0 \end{aligned}\right\} \text{Aporte al residuo de los polos de orden 2}$$

$$\mathbb{F} = 0 \Rightarrow \boxed{\begin{aligned} F_x &= 0 \\ F_y &= 0 \end{aligned}}$$

10.



b) Podemos evaluar mediante teorema del círculo:

$$W(z) = U_\infty . z + U_\infty . \frac{a^2}{z} \qquad U_\infty \in \mathbb{R}$$

$$\frac{a^2}{|z|^2} z \nearrow \quad = \boxed{U_\infty z \left(1 + \frac{a^2}{z^2}\right) = W(z)}$$

Ahora a este $W(z)$ le sumamos una circulación convención  Γ circulación positiva $\Gamma>0$

$$\boxed{W(z) = U_\infty z\left(1+\frac{a^2}{z^2}\right) + \frac{\Gamma}{2\pi i}\ln(z)}$$

Se mete este potencial mediante un vórtice lineal en el origen

$$U_\infty\left(z+\frac{a^2 z^*}{|z|^2}\right) - \frac{i\Gamma}{2\pi}\ln|z| + \frac{\Gamma}{2\pi \cancel{i}}\cancel{i}\theta$$

$$W(z) = \underbrace{U_\infty\left(\frac{z+z^*}{2}\right) + \frac{a^2 U_\infty}{|z|^2}\left(\frac{z+z^*}{2}\right) + \frac{\Gamma\theta}{2\pi}}_{\phi(z)} + i\underbrace{\left[U_\infty\left(\frac{z-z^*}{2i}\right) - \frac{U_\infty a^2}{|z|^2}\left(\frac{z-z^*}{2i}\right) - \frac{\Gamma}{2\pi}\ln|z|\right]}_{\psi(z)}$$

$$\frac{dW}{dz} = \frac{d}{dz}\left(U_\infty\left(z+\frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i}\ln(z)\right) = \mathbb{V}^*$$

$$\frac{dW}{dz} = U_\infty - U_\infty a^2\frac{1}{z^2} + \frac{\Gamma}{2\pi i}\frac{1}{z} \rightarrow \quad \frac{dW}{dz} = 0$$

$$\mathbb{V}^* = U_\infty - \frac{a^2 U_\infty}{z^2} - \frac{i\Gamma}{2\pi}\frac{1}{z} = 0 \rightarrow$$

$$z^2 U_\infty - a^2 U_\infty - \frac{i\Gamma}{2\pi}z = 0 \qquad \text{Cuadrática en } \mathbb{C}$$

$$\left(z+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} = \frac{\frac{i^2\Gamma^2}{4\pi^2} + 4U_\infty(a^2U_\infty)}{4U_\infty^2} =$$

$$\left(z+\frac{b}{2a}\right)^2 = \frac{-\Gamma^2}{16U_\infty^2\pi^2} + a^2 \leftarrow \text{Esto es } \mathbb{R} \Rightarrow \text{no hay problema con la raíz}$$

$$z = +\frac{i\Gamma}{4U_\infty\pi} \pm \sqrt{\quad}$$

$$z_{1,2} = \frac{\frac{i\Gamma}{2\pi} \pm \sqrt{\frac{-\Gamma^2}{4\pi^2} - 4U_\infty(-a^2U_\infty)}}{2U_\infty}$$

$$z_{1,2} = \frac{i\Gamma}{4\pi U_\infty} \pm \sqrt{\frac{-\Gamma^2}{U_\infty^2 4\pi^2 4} + \frac{\cancel{4}a^2\cancel{U_\infty^2}}{\cancel{4U_\infty^2}}}$$

$$z_{1,2} = \frac{i\Gamma}{4\pi U_\infty} \pm \left(\frac{-\Gamma^2}{16\pi^2U_\infty^2} + a^2\right)^{1/2}$$

Tendremos varios casos según el valor de Γ:

- $\Gamma=0 \rightarrow \boxed{z=\pm a}$ son puntos de estancamiento

$$\left.\begin{aligned}\frac{-\Gamma^2}{16\pi^2U_\infty} + a^2 &= 0\\ a^2 16\pi^2 U_\infty &= \Gamma^2\\ a\pi 4 U_\infty &= \Gamma\end{aligned}\right\} \rightarrow \bullet\ \Gamma = 4a\pi U_\infty \qquad \boxed{z = ia} \leftarrow \text{único punto de estancamiento}$$

- $\Gamma < 4a\pi U_\infty$ $\qquad \boxed{z = \frac{i\Gamma}{4\pi U_\infty} \pm \underbrace{\sqrt{a^2 - \frac{\Gamma^2}{16\pi^2U_\infty^2}}}_{\in\mathbb{R}}}$ $\leftarrow$ dos puntos de estancamiento a los lados del eje imaginario

$$\frac{-\Gamma^2}{16\pi^2U_\infty} + a^2 > 0$$
$$16\pi^2U_\infty^2a^2 > \Gamma^2$$
$$4\pi U_\infty a > \Gamma$$

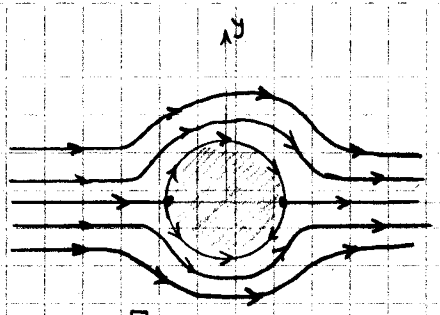
- $\Gamma > 4a\pi U_\infty$ $\qquad \boxed{z = \frac{i\Gamma}{4\pi U_\infty} \pm i\sqrt{\frac{\Gamma^2}{16\pi^2U_\infty^2} - a^2}}$ $\leftarrow$ dos puntos de estancamiento sobre el eje imaginario

$$a^2 > \frac{\Gamma^2}{4\pi U_\infty}$$

$$a^2 > \frac{\Gamma}{4\pi a} > a$$

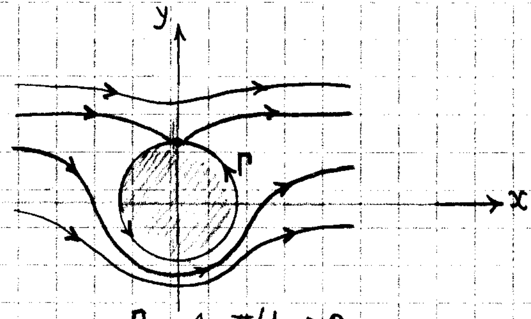
$$\cancel{a}\, a(a-1) > \frac{\Gamma^2}{4\pi U_\infty}$$

a)



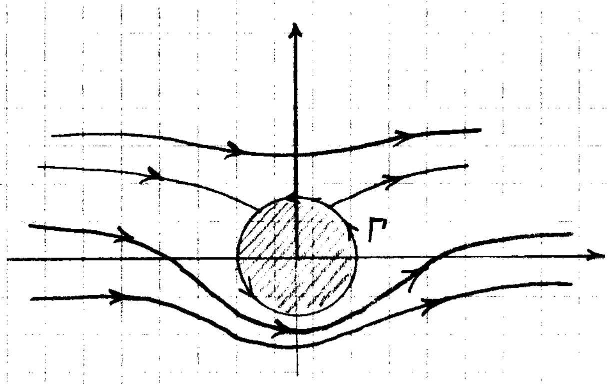
$\Gamma = 0$

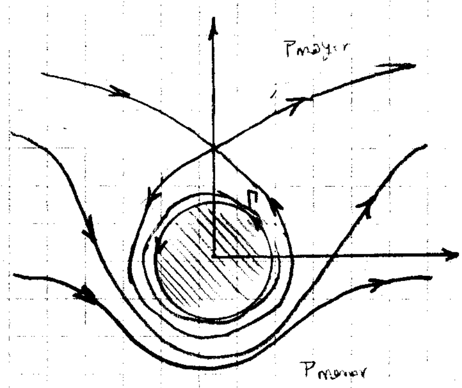
Es el caso ya resuelto; dos puntos de estancamiento diametralmente opuestos



$\Gamma = 4a\pi U_\infty > 0$

Un punto de estancamiento crítico





$\Downarrow$ empuje $-\rho\Gamma U_\infty$

Los puntos de estancamiento se han desprendido

c)

Para calcular la fuerza que hace el líquido sobre el sólido se usa el teorema de Blasius

$$\mathbb{F} = \frac{i\rho}{2}\oint\left(\frac{dW}{dz}\right)^2 dz$$

$$\frac{dW}{dz} = U_\infty - U_\infty\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z} \rightarrow$$

$$\left(\frac{dW}{dz}\right)^2 = \left(U_\infty - U_\infty\frac{a^2}{z^2}\right)^2 - \cancel{2}\,U_\infty\left(1-\frac{a^2}{z^2}\right)\frac{i\Gamma}{\cancel{2}\pi z} + \frac{i^2\Gamma^2}{4\pi^2 z^2}$$

$$= U_\infty^2 \underbrace{- 2U_\infty^2\frac{a^2}{z^2}}_{\text{polo de orden 2 en } z=0} + \underbrace{U_\infty^2\frac{a^4}{z^4}}_{\text{polo de orden 4 en } z=0} - \underbrace{\frac{U_\infty i\Gamma}{\pi z}}_{\text{polo de orden 1}} + \underbrace{\frac{U_\infty a^2 i\Gamma}{z^3\pi}}_{\text{polo de orden 3}} - \underbrace{\frac{\Gamma^2}{4\pi^2 z^2}}_{\text{polo de orden 2}}$$

De estos solo tenemos residuos no nulos de $-\frac{U_\infty i\Gamma}{\pi z}$

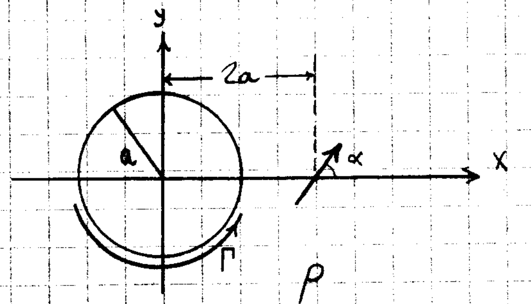
$$\text{Res}\left(\frac{dW}{dz}\right)^2 = \lim_{z\to 0} -\frac{U_\infty i\Gamma}{\pi} = -\frac{i\Gamma U_\infty}{\pi} \rightarrow$$

$$\mathbb{F} = \frac{i\rho}{\cancel{2}}\,\cancel{2}\pi i\left(-\frac{i\Gamma U_\infty}{\pi}\right) = +\rho\, i\,\Gamma U_\infty \Rightarrow F_x - iF_y = i\rho\Gamma U_\infty$$

$$\boxed{F_y = -\rho\Gamma U_\infty}$$

La fuerza se halla en la dirección correcta.

11.



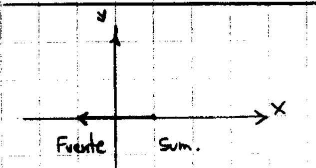
Podremos aplicar teorema del círculo para un dipolo.

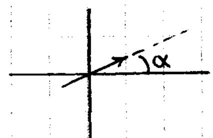
$$W(z) = -\frac{\mu . e^{i\alpha}}{2\pi}\cdot\frac{1}{z}$$ dipolo en el origen

$$W(z) = -\frac{\mu e^{i\alpha}}{2\pi}\cdot\frac{1}{(z-2a)} - \frac{\mu e^{-i\alpha}}{2\pi}\cdot\frac{1}{\left(\frac{a^2}{z}-2a\right)} + \frac{\Gamma}{2\pi i}\ln z$$

La circulación (vórtice en el origen)

Notas dipolo

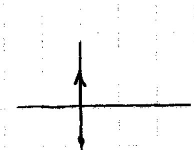




$$W(z) = \frac{\mu}{2\pi}\cdot\frac{1}{z} \longrightarrow W(z).e^{i(\pi+\alpha)} = -\frac{\mu . e^{i\alpha}}{2\pi}\cdot\frac{1}{z}$$

$$\downarrow$$

$$W(z).e^{-i\pi/2} = -\frac{\mu . i}{2\pi}\cdot\frac{1}{z}$$



Pero $\frac{1}{(a^2/z-2a)} =$

$$\left(\frac{a^2-2a.z}{z}\right)^{-1} = \frac{-z}{2az-a^2}$$

$$= \frac{-z/2a}{(z-a/2)}$$

Conviene escribir el $W(z)$ en términos de cosas sencillas.

$$+\frac{z}{2a} = +\frac{z}{2a} + \frac{1}{4} - \frac{1}{4} = +\frac{1}{2a}\left(z-\frac{a}{2}\right)+\frac{1}{4}$$

$$W(z) = -\frac{\mu}{2\pi}.e^{i\alpha}\frac{1}{(z-2a)} + \frac{\mu . e^{-i\alpha}}{2\pi}\cdot\frac{z/2a}{\left(z-\frac{a}{2}\right)} - \frac{i\Gamma}{2\pi}.\ln(z)$$

$$W(z) = -\frac{\mu e^{i\alpha}}{2\pi}\cdot\frac{1}{(z-2a)} + \underbrace{\frac{\mu . e^{-i\alpha}}{2\pi}\cdot\frac{1}{2a}}_{\text{Constante}} + \frac{\mu e^{-i\alpha}}{2\pi}\cdot\frac{1/4}{\left(z-\frac{a}{2}\right)} - \frac{i\Gamma}{2\pi}.\ln(z)$$

$$\frac{dW(z)}{dz} = \frac{\mu . e^{i\alpha}}{2\pi}\cdot\frac{1}{(z-2a)^2} - \frac{\mu . e^{-i\alpha}}{2\pi}\cdot\frac{1/4}{\left(z-\frac{a}{2}\right)^2} - \frac{i\Gamma}{2\pi}\cdot\frac{1}{z}$$

definimos $\frac{\mu . e^{i\alpha}}{2\pi} = \phi$

$$\left[\frac{dW}{dz}(z)\right]^2 = \left(\phi\cdot\frac{1}{(z-2a)^2} - \frac{\phi^*}{4}\cdot\frac{1}{\left(z-\frac{a}{2}\right)^2} - \frac{i\Gamma}{2\pi}\cdot\frac{1}{z}\right)\left(\frac{\phi}{(z-2a)^2} - \frac{\phi^*}{4\left(z-\frac{a}{2}\right)^2} - \frac{i\Gamma}{2\pi z}\right)$$

$$\left(\frac{dW}{dz}\right)^2 = \frac{\phi^2}{(z-2a)^4} + \underbrace{\frac{(\phi^*)^2}{16\left(z-\frac{a}{2}\right)^4}}_{} + \underbrace{\frac{i^2\Gamma^2}{(2\pi)^2}\cdot\frac{1}{z^2}}_{\#4}$$ → polo de orden 2 en 0; → Polo de orden 4 en $a/2$

$$-\underbrace{\frac{2|\phi|^2}{4(z-2a)^2\left(z-\frac{a}{2}\right)^2}}_{\#3} - \underbrace{\frac{2\,\phi\, i\Gamma}{2\pi(z)(z-2a)^2}}_{\text{polo de orden 1 en 0}\ \#1} + \underbrace{\frac{2\phi^* i\Gamma}{2\pi . 4 . z\left(z-\frac{a}{2}\right)^2}}_{\text{polo de orden 1 en 0}}$$

polo de orden 2 en $a/2$ ← (#3)

→ #2 polo de orden 2 en $\frac{a}{2}$

Las singularidades internas serán $z=0$, $z=\frac{a}{2}$

$$\text{Res}\{\#1\ , z=0\} = \lim_{z\to 0} -\frac{\phi i\Gamma}{\pi}\cdot\frac{1}{(z-2a)^2} = -\frac{i\Gamma\phi}{\pi 4a^2}$$

$$\text{Res}\{\#2, z=0\} = \lim_{z\to 0} + \frac{\phi^* i\Gamma}{4\pi\left(z-\frac{a}{2}\right)^2} = \frac{i\Gamma\phi^*\cancel{4}}{\cancel{4}\pi\ a^2}$$

$$\text{Res}\{\#4, z=0\} = \frac{d}{dz}\left(-\frac{\Gamma^2}{4\pi^2}\right)\Bigg|_{z=0} = 0$$

$$\text{Res}\left\{\#3,\ z=a/2\right\} = \frac{d}{dz}\left(-\frac{|\phi|^2}{2(z-2a)^2}\right)\Bigg|_{z=a/2} = -\frac{|\phi|^2}{2}\,\frac{-2}{(z-2a)^3}\Bigg|_{z=a/2} = \frac{|\phi|^2}{\left(\frac{a}{2}-2a\right)^3}$$

$$= \frac{|\phi|^2}{3^3a^3}\cdot 8 = \frac{8}{27}\cdot\frac{|\phi|^2}{a^3}$$

$$\text{Res}\left\{\#2,\ z=a/2\right\} = \frac{d}{dz}\left(\frac{+\phi^* i\Gamma}{4\pi\, z}\right)\Bigg|_{a/2} = -\frac{i\Gamma\phi^*}{4\pi}\cdot\frac{1}{z^2}\Bigg|_{z=a/2} = \frac{-i\Gamma\phi^*\cancel{4}}{\cancel{4}\pi\cdot a^2}$$

$$\sum \text{Res} = \frac{-i\Gamma\phi}{4\pi a^2} + \cancel{\frac{i\Gamma\phi^*}{\pi a^2}} + \frac{8}{27a^3}\cdot|\phi|^2 - \cancel{\frac{i\Gamma\phi^*}{\pi a^2}}$$

Ahora bien:

$$\mathbb{F} = \frac{i\rho}{2}\oint\left(\frac{dW}{dz}\right)^2 dz = \frac{i\rho}{2}\cdot 2\pi i\cdot\sum_k \text{Res}\left\{\left(\frac{dW}{dz}\right)^2, z_k\right\} \Rightarrow$$

$$\mathbb{F} = \frac{i\rho}{\cancel{2}}(\cancel{2}\pi i)\cdot\left(\frac{-i\Gamma}{4\pi a^2}\cdot\frac{\mu\cdot e^{i\alpha}}{2\pi} + \frac{8}{27a^3}\,\frac{\mu^2}{4\pi^2}\right)$$

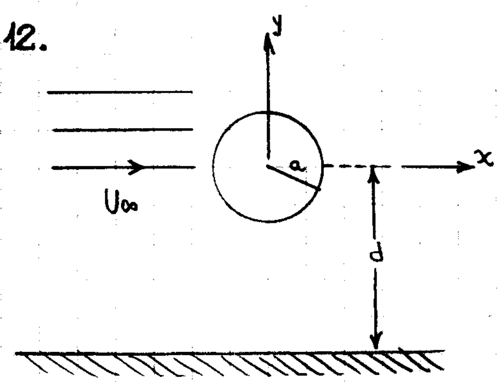
$$\mathbb{F} = -\frac{\pi\rho}{4\pi^2}\Bigg(\underbrace{\frac{-i\Gamma\mu e^{i\alpha}}{2a^2} + \frac{8}{27a^3}\mu^2}\Bigg)$$

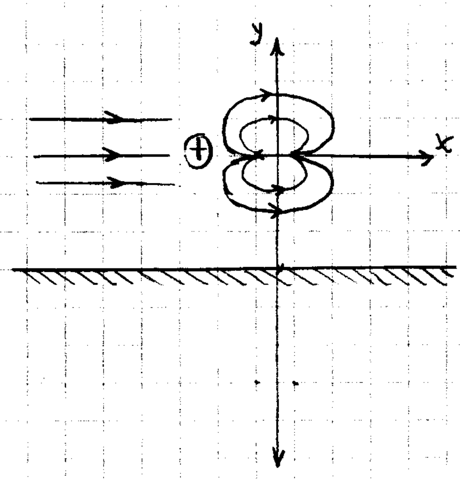
Necesito que se anule esta parte

$$\frac{i\Gamma\cancel{\mu}e^{i\alpha}}{2a^{\cancel{2}}} = \frac{8\mu^{\cancel{2}}}{27a^{\cancel{3}}}$$

$$\boxed{\mu = \frac{27\, a\cdot i\Gamma\cdot e^{i\alpha}}{16}}$$

12.





Por teorema del círculo para U_∞ y el cilindro (solos)

$$W(z) = U_\infty z + U_\infty\frac{a^2}{z} \quad , U_\infty\in\mathbb{R}$$

Es decir un flujo uniforme al infinito más un dipolo en el origen. Supongamos que el plano aparece: obviamente necesitará ser línea de corriente ⇒ pensamos en otro dipolo en $y=-2d$

$$W(z) = U_\infty z + U_\infty\frac{a^2}{z} + U_\infty\frac{a^2}{(z+i2d)}$$

funciona en un punto

$$W(z=-id) = -U_\infty id + \frac{U_\infty a^2}{-id} + U_\infty\frac{a^2}{id}$$

No funciona en la línea de corriente

$$W(z=x-id) = U_\infty(x-id) + \frac{U_\infty a^2}{(x-id)} + U_\infty\frac{a^2}{(x+id)}$$

$$= U_\infty(x-id) + \frac{U_\infty a^2(x+id)}{|x-id|^2} + \frac{U_\infty a^2(x-id)}{|x+id|^2}$$

$$U_\infty x - idU_\infty + \frac{U_\infty a^2 x\, 2}{(x^2+d^2)}$$