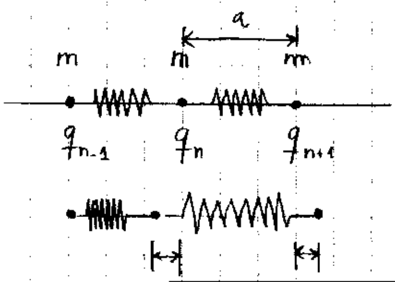
Guía 1 : Cuantificación de un sistema de partículas y Teorías de Campos Clásica

1.



(a)

$$\boxed{L = \sum_n \frac{1}{2} m \dot{q}_n^2 - \frac{k}{2}(q_{n+1} - q_n)^2}$$

$$H = \sum_i p_i \cdot \dot{q}_i - L \qquad \text{con} \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_n m \cdot \dot{q}_n^2 - \left(\sum_n \frac{1}{2} m \dot{q}_n^2 - \frac{k}{2}(q_{n+1} - q_n)^2 \right)$$

$$\boxed{H = \sum_n \frac{1}{2} m \dot{q}_n^2 + \frac{k}{2}(q_{n+1} - q_n)^2}$$

$n = i-1$
$n = i$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(m \dot{q}_i \right) - \left[+k(q_{i+1} - q_i) - k(q_i - q_{i-1}) \right]$$

$$\boxed{\sum_n m \ddot{q}_n - k(q_{n+1} - q_n) + k(q_n - q_{n-1}) = 0}$$

Se puede escribir entonces

$$ka \left(\frac{q_{n+1} - q_n}{a} \right) = ka \cdot \frac{q(x+a) - q(x)}{a} \rightarrow ka \cdot \frac{\partial q}{\partial x}\Big|_x$$

$$ka \left(\frac{q_n - q_{n-1}}{a} \right) = ka \cdot \frac{q(x) - q(x-a)}{a} \rightarrow ka \cdot \frac{q(x'+a) - q(x')}{a} = ka \cdot \frac{\partial q}{\partial x}\Big|_{x-a}$$

$x - a = x'$

$$\therefore \quad k(q_{n+1} - q_n) - k(q_n - q_{n-1}) \rightarrow ka \left[\frac{\partial q}{\partial x}\Big|_x - \frac{\partial q}{\partial x}\Big|_{x-a} \right]$$

$$\Rightarrow \quad m\ddot{q}_n + ka \left[\partial_x q|_{x-a} - \partial_x q|_x \right] = 0$$

$$\frac{m}{a}\ddot{q}_n + ka \cdot \frac{1}{a}\left(\partial_x q|_{x-a} - \partial_x q|_x \right) = 0 \rightarrow \text{cambio } a \rightarrow -a$$

$$-\frac{m}{a}\ddot{q}_n + ka \left[\frac{\partial_x q|_{x+a} - \partial_x q|_x}{a} \right] = 0$$

tiendo $\underset{n \to \infty}{a \to 0} \Rightarrow ka \to Y$, $\frac{m}{a} \to \mu$

$$\mu \frac{\partial^2 q}{\partial t^2} - Y \cdot \frac{\partial^2 q}{\partial x^2} = 0$$

$$\boxed{\partial_t^2 q - \frac{Y}{\mu} \partial_x^2 q = 0}$$

Como es una ecuación de ondas resulta que:

$$\frac{Y}{\mu} = v^2 \rightarrow \frac{Y}{\mu} = \frac{k.a}{m/a} \rightarrow \boxed{v = \sqrt{\frac{k.a^2}{m}}}$$

(b) Se pueden usar condiciones de contorno periódicas $\Rightarrow$

$$q_{N+1} = q_1$$

$$m\ddot{q}_n - k\left(q_{n+1} + q_{n-1} - 2q_n\right) = 0$$

con u_n^k base de ondas planas

si $q_n = \sum_k a_k(t) u_n^k \rightarrow \ddot{q}_n = \sum_k \ddot{a}_k(t) u_n^k \Rightarrow$

$$\sum_k m\,\ddot{a}_k(t) u_n^k - k\left[a_k(t) u_{n+1}^k + a_k(t) u_{n-1}^k - 2a_k(t) u_n^k\right] = 0$$

$$(1) \quad \sum_k \left(m\,\ddot{a}_k(t) + 2ka_k(t)\right) u_n^k - k\,a_k(t)\, u_{n+1}^k - k\,a_k(t)\, u_{n-1}^k = 0$$

Como la base es ortonormal:

$$\sum_{n=1}^{N} u_n^{k'*} \cdot u_n^k = \delta_{kk'}$$

y como es completa

$$\sum_k u_{n'}^{k*} \cdot u_n^k = \delta_{nn'}$$

$\Rightarrow$ Hacemos $\sum_n u_n^{k'*} \cdot [1]$

$$\sum_n \sum_k \left(m\,\ddot{a}_k + 2k\,a_k\right) u_n^{k'*} \cdot u_n^k - \sum_n \sum_k k a_k u_n^{k'*} \cdot u_{n+1}^k - \sum_n \sum_k k\,a_k\, u_n^{k'*} \cdot u_{n-1}^k = 0$$

$$\left(m\ddot{a}_{k'} + 2k\,a_{k'}\right) - \sum_n \sum_k k\,a_k\left(u_n^{k'*} \cdot u_{n+1}^k + u_n^{k'*} \cdot u_{n-1}^k\right) = 0$$

$$\ddot{a}_{k'} = -\frac{2\varkappa a_{k'}}{m} + \sum_n \sum_{k'} \frac{\varkappa a_{k'}}{m}\left(u_n^{k'*} \cdot u_{n+1}^k + u_n^{k'*} \cdot u_{n-1}^k\right)$$

Dado que u_n^k son ondas planas

$$u_n^k = N.e^{ik.x} \rightarrow$$

$$u_{N+1}^k = N.e^{ik(N+1)a} = N.e^{ika} = u_1^k$$

$$e^{ikNa} = 1 \Rightarrow kNa = 2\pi.l \quad , l \in \mathbb{Z}$$

$$k = \frac{2\pi}{Na} l$$

$$\sum_n u_n^{k'*} \cdot u_n^k = \sum_n e^{-ik'na} e^{ikna} . N^2$$

$$\delta_{kk'} = \sum_n e^{ina(k-k')}$$

$$\Rightarrow \delta_{kk'} = \sum_n N^2 \rightarrow N^2 = \frac{1}{N}$$

$$u_n^k = \frac{1}{\sqrt{N}} . e^{ikna} \Rightarrow u_n^{k*} = \frac{1}{\sqrt{N}} e^{-ikna}$$

$$u_{n+1}^k = \frac{1}{\sqrt{N}} e^{ikna} \cdot e^{ika}$$

$$u_{n-1}^k = \frac{1}{\sqrt{N}} \cdot e^{ikna} \cdot e^{-ika}$$

$$\ddot{a}_k = -\frac{2\varkappa a_k}{m} + \sum_{n,k'} \frac{\varkappa a_{k'}}{m} \left(\frac{1}{\sqrt{N}} e^{-ik'na} \frac{1}{\sqrt{N}} e^{ikna} \cdot e^{ika} + \right.$$
$$\left. \frac{1}{\sqrt{N}} e^{-ik'na} \frac{1}{\sqrt{N}} e^{ikna} \cdot e^{-ika} \right)$$

$$\frac{1}{N} \frac{\varkappa}{m} \sum_{n,k'} a_{k'} e^{ina(k-k')} \left(e^{ika} + e^{-ika} \right)$$

$$\ddot{a}_k = -\frac{2\varkappa a_k}{m} + \frac{\varkappa}{N.m} \left(e^{ika} + e^{-ika} \right) \sum_n \sum_{k'} a_{k'} e^{ina(k-k')}$$

$$\sum_n a_k$$

$$\ddot{a}_k = -\frac{2\varkappa a_k}{m} + \frac{\varkappa}{m} 2 \cos(ka) \cdot a_k$$

$$\ddot{a}_k(t) = \frac{\varkappa}{m} 2 \left[\cos(ka) - 1 \right] a_k = -\omega_k^2 a_k$$

con $\boxed{\omega_k = \sqrt{\frac{2\varkappa}{m} \left(1 - \cos(ka) \right)}}$

Entonces llegamos a un oscilador armónico con dispersión ω_k.

$$a_k(t) = A \cdot e^{-i\omega_k t} + B \cdot e^{i\omega_k t}$$
$$\ddot{a}_k(t) = A \cdot e^{-i\omega_k t} \cdot (-i\omega_k)^2 + B \cdot e^{i\omega_k t} (i\omega_k)^2$$
$$\ddot{a}_k(t) = A \cdot \omega_k^2 \cdot e^{-i\omega_k t} - B \cdot \omega_k^2 \cdot e^{i\omega_k t}$$

Como q_n deben ser reales

$$a_k^*(t)\, u_n^{k*} = a_{k'}(t)\, u_n^{k'}$$

$$a_k^* \frac{1}{\sqrt{N}} e^{-ikna} = a_{k'} \frac{1}{\sqrt{N}} e^{ikna} \Leftrightarrow$$

$$(-k) = k' \rightarrow$$

$$a_k^* = a_{-k} \Rightarrow$$

$$a_k^*(t) = A_k^* e^{i\omega_k t} + B_k^* e^{-i\omega_k t}$$
$$a_{-k}(t) = A_{-k} e^{-i\omega_k t} + B_{-k} e^{+i\omega_k t} \qquad \text{pues } \omega_k = \omega_{-k}$$

$$B_k^* = A_{-k} \wedge A_k^* = B_{-k} \rightarrow$$

$$b_R^{(k)} - i b_I^{(k)} = a_R^{(-k)} + i a_I^{(-k)} \rightarrow a_R^{(-k)} = b_R^{(k)} \wedge b_I^{(k)} = -a_I^{(-k)}$$
$$b_R^{-k} + i b_I^{(-k)} = a_R^{(k)} - i a_I^{(k)} \rightarrow a_I^{k} = -b_I^{-k} \quad a_R^{(k)} = b_R^{(-k)}$$

$$a_k(t) = (b_R^{-k} - b_I^{-k}) e^{-i\omega_k t} + (b_R^{k} + i b_I^{k}) e^{i\omega_k t}$$

$$q_n(t) = \sum_k \left(b_{-k}^* e^{-i\omega_k t} + b_k \cdot e^{i\omega_k t} \right) u_n^k$$

$$\boxed{q_n(t) = \frac{1}{\sqrt{N}} \sum_k \left(b_{-k}^* \cdot e^{-i\omega_k t + ikna} + b_k \cdot e^{i\omega_k t + ikna} \right)}$$

Para el cálculo del hamiltoniano separamos en $H = T + V$

$$T = \sum_n \frac{m}{2} \dot{q}_n^2 \Rightarrow$$

$$\dot{q}_n = \sum_k \left[b_{-k}^* \cdot e^{-i\omega_k t} (-i\omega_k) + b_k e^{i\omega_k t} (i\omega_k) \right] u_n^k$$

$$(\dot{q}_n)^2 = \sum_{k,k'} \left[-b_{-k}^* e^{-i\omega_k t} i\omega_k + i\omega_k b_k e^{i\omega_k t} \right] u_n^k u_n^{k'}$$
$$\left[-b_{-k'}^* e^{-i\omega_{k'} t} i\omega_{k'} + i\omega_{k'} b_{k'} e^{i\omega_{k'} t} \right]$$

$$T = \sum_n \frac{m}{2} \sum_k \sum_{k'} u_n^k u_n^{k'} \left[+ b_{-k}^* b_{-k'}^* e^{-it(\omega_k + \omega_{k'})} (-1) \omega_k \omega_{k'} - \omega_k \omega_{k'} b_k b_{k'} e^{it(\omega_k + \omega_{k'})} \right.$$
$$\left. + \omega_k b_{-k}^* b_{k'} \omega_{k'} e^{-it(\omega_k - \omega_{k'})} + \omega_k \omega_{k'} b_k b_{-k'}^* e^{+it(\omega_k - \omega_{k'})} \right]$$

$$T = \sum_k \frac{m}{2} \left(-b_{-k}^* b_k^* e^{-i2\omega_k t} \omega_k^2 - b_k b_{-k} e^{+i2\omega_k t} \omega_k^2 + b_{-k}^* b_{-k} \omega_k^2 + b_k b_k^* \omega_k^2 \right)$$

$$T = \sum_k \frac{m}{2} (-\omega_k^2) \left[b_{-k}^* b_k^* e^{-i2\omega_k t} + b_k b_{-k} e^{i2\omega_k t} - b_{-k}^* b_{-k} - b_k b_k^* \right]$$

$$V = \sum_n \frac{\varkappa}{2} (q_{n+1} - q_n)^2 \Rightarrow$$

$$q_{n+1} - q_n = \sum_k \left(b_{-k}^* e^{-i\omega_k t} + b_k e^{i\omega_k t} \right) \left(u_{n+1}^k - u_n^k \right)$$
$$= \sum_k \left(b_{-k}^* e^{-i\omega_k t} + b_k e^{i\omega_k t} \right) (e^{ika} - 1) u_n^k$$

$$(q_{n+1} - q_n) = \sum_k b_k^* e^{-i\omega_k t} [e^{-ika} - 1] u_n^{-k} + b_k e^{i\omega_k t} [e^{ika} - 1] u_n^k$$

¡pero

$$u_n^k = \frac{1}{\sqrt{N}} e^{ikn}, \quad u_n^{k*} = \frac{1}{\sqrt{N}} e^{-ikn} = u_n^{-k}$$

$$(q_{n+1} - q_n)^2 = \sum_{k'} \sum_k b_k^* b_{k'}^* e^{-i\omega_k t} e^{-i\omega_{k'} t} [e^{-ika} - 1][e^{-ik'a} - 1] u_n^{k*} u_n^{k'*}$$
$$+ b_k b_{k'} e^{i\omega_k t} e^{i\omega_{k'} t} [e^{ika} - 1][e^{ik'a} - 1] u_n^k u_n^{k'}$$
$$+ b_k^* b_{k'} e^{-i\omega_k t} e^{i\omega_{k'} t} [e^{-ika} - 1][e^{ik'a} - 1] u_n^{k*} u_n^{k'}$$
$$+ b_k b_{k'}^* e^{i\omega_k t} e^{-i\omega_{k'} t} [e^{ika} - 1][e^{-ik'a} - 1] u_n^k u_n^{k'*}$$

$$\sum_n \frac{\varkappa}{2}(q_{n+1}-q)^2 = \sum_k \frac{\varkappa}{2}\Big[b_k^* b_{-k}^* e^{-i2\omega_k t}(e^{-ika}-1)(e^{ika}-1)$$
$$+ b_k b_{-k} e^{i2\omega_k t}(e^{ika}-1)(e^{+ika}-1)$$
$$+ b_k^* b_k (e^{-ika}-1)(e^{ika}-1)$$
$$+ b_k b_k^* (e^{-ika}-1)(e^{ika}-1)\Big]$$
$$= \sum_k \frac{\varkappa}{2}(e^{-ika}-1)(e^{ika}-1)\Big[b_k^* b_{-k}^* e^{-i2\omega_k t} + b_k b_{-k} e^{i2\omega_k t} + b_k^* b_k + b_k b_k^* \Big]$$

Pero resulta que:

$$(e^{-ika}-1)(e^{ika}-1)$$
$$1 - e^{ika} - e^{-ika} + 1$$
$$2 - 2\cos(ka)$$
$$2(1-\cos(ka))$$

$$V = \frac{m}{2}\sum_k \underbrace{\frac{\varkappa}{m}2(1-\cos(ka))}_{\omega_k^2}\Big[b_k^* b_{-k}^* e^{-i2\omega_k t} + b_k b_{-k} e^{i2\omega_k t} + 2 b_k b_k^* \Big]$$

Sumando es:

$$H = T+V = \sum_k -\frac{m\omega_k^2}{2}\Big(\cancel{b_{-k}^* b_k^* e^{-i2\omega_k t}} + \cancel{b_k b_{-k} e^{i2\omega_k t}} - b_{-k}^* b_{-k} - b_k b_k^* - \cancel{b_k^* b_{-k}^* e^{i2\omega_k t}} - \cancel{b_k b_{-k} e^{+i2\omega_k t}} - 2 b_k b_k^* \Big)$$

$$H = \sum_k -\frac{m\omega_k^2}{2}\left(-b_k^* b_k 4\right)$$

$$\boxed{H = \sum_k 2m\,\omega_k^2\, b_k^* b_k}$$

AUXILIAR

$$\sum_{n=1}^{N} u_n^{k*} u_n^{k'} = \delta_{kk'}$$
$$\frac{1}{\sqrt{N}}e^{-ikn}\,\frac{1}{\sqrt{N}}e^{ik'n} = \frac{1}{N}e^{in(k'-k)} = \begin{cases}0\\1\end{cases}$$

$$\sum_{n=1}^{N} u_n^{k*} u_n^{k*} = \delta_{k',-k}$$
$$\frac{1}{\sqrt{N}}e^{-ik'n}\,\frac{1}{\sqrt{N}}e^{-ikn} = \frac{1}{N}e^{-in(k+k')} = \begin{cases}0\\1 \text{ si } k'=-k\end{cases}$$

2. Para cuantificar pasamos a operadores según:

$$q_n \to \hat{q}_n \quad \text{con} \quad [\hat{q}_n, \hat{q}_{n'}] = 0$$
$$p_n \to \hat{p}_n \qquad [\hat{p}_n, \hat{p}_{n'}] = 0$$
$$[\hat{q}_n, \hat{p}_{n'}] = +i\hbar\,\delta_{nn'}$$

$$[\hat{q}_n, \hat{p}_{n'}] \doteq q_n\left(-i\hbar \frac{\partial}{\partial q_{n'}}\Psi\right) + i\hbar\frac{\partial}{\partial q_{n'}}(q_n\Psi)$$
$$-i\hbar\, q_n \frac{\partial\Psi}{\partial q_{n'}} + i\hbar\,\Psi\delta_{nn'} + i\hbar\, q_n \frac{\partial\Psi}{\partial q_{n'}} = +i\hbar\,\delta_{nn'}\Psi$$

$$\boxed{\hat{q}_n(t) = \frac{1}{\sqrt{N}}\sum_k \left(\hat{b}^+_{-k}\, e^{-i\omega_k t} + \hat{b}_k\, e^{i\omega_k t}\right) u_n^k}$$

¡porque la descomposición en modos normales es simil a la del caso clásico. Veamos que $\hat{q}_n^+ = \hat{q}_n$

$$\hat{q}^+ = \frac{1}{\sqrt{N}}\sum_k \left(\hat{b}_{-k}\, e^{i\omega_k t} + \hat{b}_k^+ e^{-i\omega_k t}\right) u_n^{k*}$$
$$\hat{q}^+ = \frac{1}{\sqrt{N}}\sum_k \left(\hat{b}_{-k}\, e^{i\omega_k t} + \hat{b}_k^+ e^{-i\omega_k t}\right) u_n^{-k}$$
$$\hat{q}^+ = \frac{1}{\sqrt{N}}\sum_k \left(\hat{b}_k\, e^{i\omega_k t} u_n^k + \hat{b}_{-k}^+ e^{-i\omega_k t} u_n^k\right) = \hat{q}_n$$

Ahora buscamos hallar $\hat{b}_k$

$$\sum_n u_n^{k'*}\hat{q}_n = \frac{1}{\sqrt{N}}\sum_k\sum_n \left(\hat{b}_{-k} e^{i\omega_k t} + \hat{b}_k^+ e^{-i\omega_k t}\right) u_n^{k'*} u_n^k$$
$$= \frac{1}{\sqrt{N}}\sum_n \left(\hat{b}_{-k'} e^{i\omega_k t} + \hat{b}_{k'} e^{-i\omega_k t}\right)$$

$$\sum_n u_n^{k'*}\hat{q}_n = \hat{b}_{-k}\, e^{i\omega_k t} + \hat{b}_k\, e^{-i\omega_k t}$$

$$\dot{\hat{q}}_n(t) = \frac{1}{\sqrt{N}}\sum_k \left(\hat{b}_{-k}\, e^{i\omega_k t}(i\omega_k) + \hat{b}_k\, e^{-i\omega_k t}(-i\omega_k)\right) u_n^k$$
$$= \frac{1}{\sqrt{N}}\sum_k i\omega_k \left(\hat{b}_{-k}\, e^{i\omega_k t} - \hat{b}_k\, e^{-i\omega_k t}\right) u_n^k$$

$$\sum_n u_n^{k'*}\dot{\hat{q}}_n = \frac{1}{\sqrt{N}}\sum_k\sum_n i\omega_k \left(\hat{b}_{-k}\, e^{i\omega_k t} - \hat{b}_k\, e^{-i\omega_k t}\right) u_n^{k'*} u_n^k$$

$$\sum_n u_n^{k'*}\dot{\hat{q}}_n = i\omega_k \left(\hat{b}_{-k}\, e^{i\omega_k t} - \hat{b}_k\, e^{-i\omega_k t}\right)$$

Pero como: $\quad p_n = m\dot{q}_n \Rightarrow \quad \hat{p}_n = m\dot{\hat{q}}_n \Rightarrow$

$$\sum_n u_n^{k'*}\hat{p}_n = im\omega_k \left(\hat{b}_{-k}\, e^{i\omega_k t} - \hat{b}_k\, e^{-i\omega_k t}\right)$$

$$\sum_n u_n^{k*} \cdot \hat{q}_n = \hat{b}_{-k} \cdot e^{i\omega_k t} + \hat{b}_k \cdot e^{-i\omega_k t} \qquad \Rightarrow$$

$$\frac{1}{i\omega_k m} \sum_n u_n^{k'*} \cdot \hat{p}_n = \hat{b}_{-k} e^{i\omega_k t} - \hat{b}_k e^{-i\omega_k t}$$

$$\frac{1}{2} \sum_n u_n^{k'*} \left(\hat{q}_n - \frac{1}{im\omega_k} \hat{p}_n \right) = \hat{b}_k \cdot e^{-i\omega_k t}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{b}_k(t) &= e^{-i\omega_k t}\, \hat{b}_k(t=0) \\ \hat{b}_k^+(t) &= e^{i\omega_k t}\, \hat{b}_k^+(t=0) \end{aligned}}$$

Copiando lo que se hizo en el ejercicio 1 y considerando ahora que b_k son operadores tenemos:

$$\hat{T} = \sum_k -\frac{m}{2}\omega_k^2 \left(\hat{b}_{-k}^+ \hat{b}_k^+ e^{-i2\omega_k t} + \hat{b}_k \hat{b}_{-k} e^{i2\omega_k t} - \hat{b}_{-k}^+ \hat{b}_{-k} - \hat{b}_k \hat{b}_k^+ \right)$$

$$\hat{V} = \sum_k \frac{m}{2}\omega_k^2 \left(\hat{b}_k^+ \hat{b}_{-k}^+ e^{-i2\omega_k t} + \hat{b}_k \hat{b}_{-k} e^{-i2\omega_k t} + \hat{b}_k^+ \hat{b}_k + \hat{b}_k \hat{b}_k^+ \right)$$

$$\hat{H} = \hat{T} + \hat{V}$$

$$= \sum_k \frac{m\omega_k^2}{\cancel{2}} \cancel{2} \left(\hat{b}_k^+ \hat{b}_k + \hat{b}_k \hat{b}_k^+ \right)$$

$$\boxed{\hat{H} = \sum_k m\omega_k^2 \left(\hat{b}_k^+ \hat{b}_k + \hat{b}_k \hat{b}_k^+ \right)}$$

Dado que el conmutador, a $t=0$, es

$$[\hat{b}_{k'}, \hat{b}_k^+] = \frac{1}{2} \sum_n u_n^{k'*} \left(\hat{q}_n + \frac{i}{m\omega_{k'}} \hat{p}_n \right) \sum_m \frac{1}{2} u_m^k \left(\hat{q}_m^+ - \frac{i}{m\omega_k} \hat{p}_m^+ \right)$$

$$- \frac{1}{2} \sum_m u_m^k \left(\hat{q}_m^+ - \frac{i}{m\omega_k} \hat{p}_m^+ \right) \sum_n \frac{1}{2} u_n^{k*} \left(\hat{q}_n + \frac{i}{m\omega_{k'}} \hat{p}_n \right)$$

$$= \sum_{n,m} \frac{1}{4} u_n^{k'*} u_m^k \left[\hat{q}_n \hat{q}_m^+ + \frac{i}{m\omega_{k'}} \hat{p}_n \hat{q}_m^+ - \frac{i}{m\omega_k} \hat{q}_n \hat{p}_m^+ - \frac{i^2}{m^2\omega_k\omega_{k'}} \hat{p}_n \hat{p}_m^+ \right]$$

$$- \frac{1}{4} u_m^k u_n^{k'*} \left[\hat{q}_m^+ \hat{q}_n - \frac{i}{m\omega_k} \hat{p}_m^+ \hat{q}_n + \frac{i}{m\omega_{k'}} \hat{q}_m \hat{p}_n - \frac{i^2}{m^2\omega_k\omega_{k'}} \hat{p}_m^+ \hat{p}_n \right]$$

$$= \sum_n \sum_m \frac{1}{4} u_n^{k'*} \cdot u_m^k \left[[\hat{q}_n, \hat{q}_m] + \frac{-i}{m\omega_k} [\hat{q}_n, \hat{p}_m] + \frac{i}{m\omega_{k'}} [\hat{p}_n, \hat{q}_m] + \frac{1}{m^2\omega_k\omega_{k'}} [\hat{p}_n, \hat{p}_m] \right]$$

$$= \sum_n \sum_m \frac{1}{4} u_n^{k'*} \cdot u_m^k \left[\frac{\hbar \delta_{nm}}{m\omega_k} + \frac{\hbar}{m\omega_{k'}} \delta_{nm} \right]$$

$$[\hat{b}_{k'}, \hat{b}_k^+] = \sum_n \frac{1}{2} \left(\frac{+\hbar}{m\omega_k} \right) u_n^{k'*} \cdot u_n^k = \frac{\hbar \delta_{kk'}}{2m\omega_k}$$

Usando esta expresión en el $\hat{H}$ se tiene

$$\hat{H} = \sum_k m\omega_k^2 \left(\hat{b}_k^+ \hat{b}_k + \frac{\hbar}{2m\omega_k} + \hat{b}_k^+ \hat{b}_k \right)$$

$$\left(2 b_k^+ b_k + \frac{1}{2} \cdot \frac{\hbar}{m\omega_k} \right)$$

$$\hat{H} = \sum_k \frac{\cancel{m}\omega_k^{\cancel{2}}\hbar}{\cancel{m\omega_k}} \left(\frac{2m\omega_k}{\hbar} \hat{b}_k^+ \hat{b}_k + \frac{1}{2} \right)$$

y podemos redefinir unos operadores $\hat{C}_k, \hat{C}_k^+$ del modo:

$$\hat{b}_{k'} \hat{b}_k^+ - \hat{b}_k^+ \hat{b}_{k'} = \frac{\hbar}{2m\omega_k} \delta_{kk'}$$

$$\left[\sqrt{\frac{2m\omega_k}{\hbar}} \hat{b}_{k'}, \sqrt{\frac{2m\omega_k}{\hbar}} \hat{b}_k^+ \right] = \delta_{kk'}$$

$$\left[\hat{C}_{k'}, \hat{C}_k^+ \right] = \delta_{kk'} \quad \Rightarrow$$

$$\boxed{\hat{H} = \sum_k \hbar\omega_k \left[\hat{C}_k^+ \hat{C}_k + \frac{1}{2} \right]}$$

De esta forma el $\hat{H}$ queda en analogía total con el hamiltoniano del oscilador armónico con lo cual sus autoestados serán:

$$|n\rangle \equiv |n_1\rangle |n_2\rangle \dots | \rangle = \prod_k |n_k\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left(\hat{C}_k^+ \right)^{n_k} |0_k\rangle$$

, donde 'n_k' es el # de partículas en el estado energético k-ésimo. Es claro que

$$\hat{C}_k^+ \hat{C}_k |n_k\rangle = \hat{C}_k^+ \sqrt{n_k} |n_k - 1\rangle = \sqrt{n_k} \hat{C}_k^+ |n_k - 1\rangle$$

$$\hat{C}_k^+ \hat{C}_k |n_k\rangle = \sqrt{n_k} \sqrt{n_k} |n_k\rangle \; n_k |n_k\rangle$$

$\Rightarrow$

$$\hat{H} |n\rangle = \sum_k \hbar\omega_k \left(n_k + \frac{1}{2} \right) |n\rangle \quad \therefore$$

$$E = \sum_k E_k \quad \text{con} \quad \boxed{E_k = \hbar\omega_k \left(n_k + \frac{1}{2} \right)}$$

3. $a|\lambda\rangle = \lambda|\lambda\rangle$ estados coherentes de $\hat{a}$ con $\lambda \in \mathbb{C}$

(a) Queremos ver que $|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^+}|0\rangle$ es coherente

$$\hat{a}|\lambda\rangle = e^{-|\lambda|^2/2} a e^{\lambda \hat{a}^+}|0\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} a \frac{\lambda^n (a^+)^n}{n!}|0\rangle$$
$$= e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a(a^+)^n|0\rangle$$

sabemos que $[a,a^+] = aa^+ - a^+a = 1 \rightarrow aa^+ = 1 + a^+a$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Conviene demostrar lo siguiente

$$[A,B^n] = [A,B^{n-1}B] = B^{n-1}[A,B] + [A,B^{n-1}]B$$
$$= B^{n-1}[A,B] + B^{n-2}[A,B]B + [A,B^{n-2}]B^2$$
$$= B^{n-1}[A,B] + B^{n-2}[A,B]B + B^{n-3}[A,B]B^2 + [A,B^{n-3}]B^3$$
$$[A,B^n] = \sum_{i=1}^{n} B^{n-i}[A,B]B^{i-1} \Rightarrow$$

$$\hat{a}|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\left(a(a^+)^n - (a^+)^n a\right)|0\rangle$$
$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{i=1}^{n} (a^+)^{n-i}[a,a^+](a^+)^{i-1}|0\rangle$$
$$e^{-|\lambda|^2/2}\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{i=1}^{n} (a^+)^{n-1}|0\rangle$$

el término $n=0$ no contribuye

$$\sum_{n=1}^{\infty} \frac{\lambda^n \cancel{(n)}}{\cancel{n}(n-1)!}(a^+)^{n-1}|0\rangle$$

$n-1 = j$

$$\hat{a}|\lambda\rangle = e^{-|\lambda|^2/2}\lambda \sum_{j=0}^{\infty} \frac{\lambda^j (a^+)^j}{j!}|0\rangle$$

$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow$ $\boxed{|\lambda\rangle \text{ es un estado coherente}}$

$$\langle\lambda| = \langle 0| e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^{n*}(a)^n}{n!} \rightarrow \langle\lambda| = \langle 0|e^{-|\lambda|^2/2} e^{\lambda^* a}$$

$$\langle\lambda|\lambda\rangle = \langle 0|e^{-|\lambda|^2/2} e^{\lambda^* a} e^{-|\lambda|^2/2} e^{\lambda a^+}|0\rangle$$
$$= e^{-|\lambda|^2}\langle 0|e^{\lambda^* a} e^{\lambda a^+}|0\rangle$$
$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\langle 0|\left(\sum_m \frac{\lambda^{*m} a^m}{m!}\right)\left(\sum_n \frac{\lambda^n (a^+)^n}{n!}\right)|0\rangle$$
$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\sum_m\sum_n \frac{\lambda^{*m}}{\sqrt{m!}}\frac{\langle 0|a^m}{\sqrt{m!}} \cdot \frac{\lambda^n}{\sqrt{n!}}\frac{(a^+)^n|0\rangle}{n!} = e^{-|\lambda|^2}\sum_{m,n} \frac{\lambda^{*m}\langle m|}{\sqrt{m!}}\frac{\lambda^n|n\rangle}{\sqrt{n!}}$$

$$e^{-|\lambda|^2}\sum_{m,n}\frac{\lambda^{*m}\lambda^n}{\sqrt{m!}\sqrt{n!}}\langle m|n\rangle = e^{-|\lambda|^2}\sum_m\sum_n\frac{\lambda^{*m}\lambda^n}{\sqrt{m!}\sqrt{n!}}\,\delta_{mn}$$

$$= e^{-|\lambda|^2}\sum_m\frac{|\lambda|^{2m}}{m!} = e^{-|\lambda|^2}e^{|\lambda|^2} = 1$$

$$\Rightarrow \boxed{|\lambda\rangle \text{ está normalizado}}$$

4.

$\phi(x,t)$ campo escalar

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi\phi \quad , \quad L(t) = \int \mathcal{L} \, d^3x$$

(a)

$$\begin{aligned} x &\xrightarrow{\text{al caso discreto}} \text{índice 'i'} \\ \phi(x,t) &\longrightarrow q_i \; : \text{coord. gen. de la masa i-ésima} \\ \int d^3x &\longrightarrow \sum_i \end{aligned}$$

(b) tiene infinitos grados de libertad dadas por los infinitos valores de $\phi(\vec{x})$ en cada punto $\vec{x}$ del dominio en cuestión.

(c)

$$L = \sum_i L_i$$

Se da este caso cuando las partículas no interactúan entre sí. EL L es suma de L_i de "UNA SOLA PARTÍCULA"

No debemos confundir esta situación con el L del ejercicio 1 donde teníamos

$$L = \sum_i \frac{1}{2} m \dot{q}_i^2 - \frac{k}{2}(q_{i+1} - q_i)^2 = \sum_i L_i \; ,$$

pero para cada partícula 'i' tenemos un término que la acopla con la partícula 'i+1', con lo cual la notación $\sum_i L_i$ es poco menos que engañosa. En el caso de $n=3$ se ve perfectamente que no podemos descomponer en Lagrangianos individuales.

(d) Una teoría de campos es local cuando la $\mathcal{L}$ solo depende de $\vec{x}$ explícitamente.

Es decir que en cada punto $\vec{x}$ del espacio, sólo depende de ese punto y NO de otro $\vec{y}$ ubicado a distancia finita.

$$\mathcal{L}_{LOCAL} = \mathcal{L}(\vec{x}) \neq \mathcal{L}(\vec{x}, \vec{y})$$

En el caso discreto un L que es NO interactuante es tal que

$$L = \sum_i L(i) \longrightarrow \quad L = \int d^3_x \, \mathcal{L}(\vec{x})$$

5.

$$\mathcal{L} = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi \phi^*$$ lagrangiana de KG

(a)

$$\frac{\partial}{\partial^\mu}\left[\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$ ← Ecuaciones de E-L para los campos

Para ϕ es

$$\frac{\partial}{\partial^\mu}\left[\partial^\mu \phi^*\right] - \left(-m^2 \phi^*\right) = 0$$

$$\partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$$

conjugando

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

Para ϕ^* es

$$\frac{\partial}{\partial^\mu}\left[\frac{\partial}{\partial[\partial_\mu \phi^*]}\left(\partial_\alpha \phi\, \eta^{\alpha\nu} \partial_\nu \phi^*\right)\right] - \left(-m^2 \phi\right) = 0$$

$$\partial_\mu \left(\partial_\alpha \phi\, \eta^{\alpha\nu} \delta^\mu_\nu\right) + m^2 \phi = 0$$

$$\leftarrow \boxed{\partial_\nu \partial^\nu \phi + m^2 \phi = 0}$$

(b) $\phi = \phi^*$ $\quad m = 0 \quad$ transformación $\phi \to \phi + \alpha \Rightarrow$

$$\partial_\mu(\phi + \alpha) = \partial_\mu \phi$$

$$J^\nu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \delta \phi_i$$ (se suma en el # de campos)

$$J^\nu = \frac{\partial}{\partial[\partial_\nu \phi]}\left(\partial_\mu \phi\, \partial^\mu \phi\right)\alpha = \left(\delta^\nu_\mu\, \partial^\mu \phi + \partial_\mu \phi\, \eta^{\mu\alpha} \delta^\nu_\alpha\right)\alpha$$

$$J^\nu = \left(\partial^\nu \phi + \partial^\nu \phi\right)\alpha$$

$$\left[\partial^\mu \phi = \eta^{\mu\alpha} \partial_\alpha \phi\right] \quad \boxed{J^\nu = \alpha . 2 . \partial^\nu \phi}$$ ← La corriente conservada en KG

(c)

$$\begin{cases} \phi \to e^{i\alpha} \phi \\ \phi^* \to e^{-i\alpha} \phi^* \end{cases} \Rightarrow \quad \phi \to \phi' \cong (1 + i\alpha)\phi$$

$$\delta\phi = \phi' - \phi = i\alpha \phi$$
$$\delta\phi^* = \phi'^* - \phi^* = -i\alpha \phi^*$$

Vale esta expresión porque solo transforman los campos

$$J_\nu = \frac{\partial \mathcal{L}}{\partial[\partial_\nu \phi]} \delta\phi + \frac{\partial \mathcal{L}}{\partial[\partial_\nu \phi^*]} \delta\phi^*$$

$$\partial^\nu \phi^* (i\alpha \phi) + \frac{\partial}{\partial[\partial_\nu \phi^*]}\left(\partial_\mu \phi\, \eta^{\mu\sigma} \partial_\sigma \phi^* - m^2 \phi \phi^*\right) \delta\phi^*$$

$$i\alpha\, \partial^\nu \phi^* . \phi - \partial_\mu \phi\, \eta^{\mu\nu} (i\alpha \phi^*)$$

$$\boxed{J_\nu = i\alpha\left[\phi^* \partial^\nu \phi - \phi\, \partial^\nu \phi^*\right]}$$ ← la corriente conservada

$$\partial^\nu J_\nu = 0 \Rightarrow 0 = \int d^3x\, \frac{1}{c}\frac{\partial}{\partial t} J_0 = \frac{1}{c}\frac{d}{dt}\int d^3x\, J_0 \to \boxed{Q \equiv \int d^3x\, J_0}$$ se conserva.

, donde: $J_0 = (i\alpha)\left(\phi^* \frac{1}{c}\frac{\partial}{\partial t}\phi - \phi \frac{1}{c}\frac{\partial}{\partial t}\phi^*\right) = \frac{i\alpha}{c}\left[\phi^* . \dot\phi - \phi . \dot\phi^*\right]$

(d) Es invariante Lorentz. Se construyen así para que la teoría sea relativista
Porque está construido con un escalar de Lorentz $\partial_\mu \phi\, \partial^\mu \phi^*$

6.

$$L_I = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad \text{con } F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Los campos son A^μ; luego la ecuación de Euler-Lagrange será:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\nu A_\mu]} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

$$(-4) \frac{\partial \mathcal{L}}{\partial [\partial_\nu A_\mu]} = \frac{\partial}{\partial [\partial_\nu A_\mu]} \left([\partial^\rho A^\alpha - \partial^\alpha A^\rho][\partial_\rho A_\alpha - \partial_\alpha A_\rho] \right.$$

$$= \frac{\partial}{\partial [\partial_\nu A_\mu]} \left[\partial^\rho A^\alpha \partial_\rho A_\alpha - \partial^\alpha A^\rho \partial_\rho A_\alpha - \partial^\rho A^\alpha \partial_\alpha A_\rho + \partial^\alpha A^\rho \partial_\alpha A_\rho \right]$$

$$= \frac{\partial}{\partial [\partial_\nu A_\mu]} \left[\eta^{\rho\beta} \partial_\beta \eta^{\alpha\gamma} A_\gamma \partial_\rho A_\alpha - \eta^{\alpha\beta} \partial_\beta \eta^{\rho\gamma} A_\gamma \partial_\rho A_\alpha - \eta^{\rho\beta} \partial_\beta \eta^{\alpha\gamma} A_\gamma \partial_\alpha A_\rho + \eta^{\alpha\beta} \partial_\beta \eta^{\rho\gamma} A_\gamma \partial_\alpha A_\rho \right]$$

$$= \left(\eta^{\rho\beta} \eta^{\alpha\gamma} - \eta^{\alpha\beta} \eta^{\rho\gamma} \right) \left[\delta_\beta^\nu \delta_\gamma^\mu \partial_\rho A_\alpha + \partial_\beta A_\gamma \delta_\rho^\nu \delta_\alpha^\mu \right]$$

$$- \left(\eta^{\rho\beta} \eta^{\alpha\gamma} - \eta^{\alpha\beta} \eta^{\rho\gamma} \right) \left[\delta_\beta^\nu \delta_\gamma^\mu \partial_\alpha A_\rho + \partial_\beta A_\gamma \delta_\alpha^\nu \delta_\rho^\mu \right]$$

$$= \eta^{\rho\nu} \eta^{\alpha\mu} \partial_\rho A_\alpha - \eta^{\alpha\nu} \eta^{\rho\mu} \partial_\rho A_\alpha + \eta^{\nu\beta} \eta^{\mu\gamma} \partial_\beta A_\gamma - \eta^{\mu\beta} \eta^{\nu\gamma} \partial_\beta A_\gamma$$
$$- \left(\eta^{\rho\nu} \eta^{\alpha\mu} \partial_\alpha A_\rho - \eta^{\alpha\nu} \eta^{\rho\mu} \partial_\alpha A_\rho + \eta^{\mu\beta} \eta^{\nu\gamma} \partial_\beta A_\gamma - \eta^{\nu\beta} \eta^{\mu\gamma} \partial_\beta A_\gamma \right)$$

$$= \partial^\nu A^\mu - \partial^\mu A^\nu + \partial^\nu A^\mu - \partial^\mu A^\nu - \left(\partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu \right)$$

$$(-4) \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} = 4 \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = 4 F^{\nu\mu} \Rightarrow \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} = -F^{\nu\mu} = F^{\mu\nu}$$

$$\partial_\nu \left(-F^{\nu\mu} \right) = 0 \quad \Rightarrow \quad \boxed{\partial_\nu F^{\nu\mu} = 0}$$

* Esta cuenta puede hacerse más fácil (como siempre sucede) tomando:

$$\frac{\partial}{\partial (\partial_\gamma A_\xi)} \left(F^{\mu\nu} F_{\mu\nu} \right) = \frac{\partial F^{\mu\nu}}{\partial (\partial_\gamma A_\xi)} F_{\mu\nu} + F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial (\partial_\gamma A_\xi)}$$

$$= \frac{\partial}{\partial (\partial_\gamma A_\xi)} \left[\partial^\mu A^\nu - \partial^\nu A^\mu \right] F_{\mu\nu} + F^{\mu\nu} \frac{\partial}{\partial (\partial_\gamma A_\xi)} \left[\partial_\mu A_\nu - \partial_\nu A_\mu \right]$$

$$= \frac{\partial}{\partial (\partial_\gamma A_\xi)} \left[\eta^{\mu\alpha} \eta^{\nu\beta} \partial_\alpha A_\beta - \eta^{\nu\alpha} \eta^{\mu\beta} \partial_\alpha A_\beta \right] F_{\mu\nu} + F^{\mu\nu} \left(\delta_\mu^\gamma \delta_\nu^\xi - \delta_\nu^\gamma \delta_\mu^\xi \right)$$

$$= \left[\eta^{\mu\alpha} \eta^{\nu\beta} \left(\delta_\alpha^\gamma \delta_\beta^\xi \right) - \eta^{\nu\alpha} \eta^{\mu\beta} \delta_\alpha^\gamma \delta_\beta^\xi \right] F_{\mu\nu} + F^{\mu\nu}$$

$$= \left[\eta^{\mu\gamma} \eta^{\nu\xi} - \eta^{\nu\gamma} \eta^{\mu\xi} \right] F_{\mu\nu} + F^{\mu\nu} \left[\delta_\mu^\gamma \delta_\nu^\xi - \delta_\nu^\gamma \delta_\mu^\xi \right]$$

$$= \eta^{\mu\gamma} \eta^{\nu\xi} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - \eta^{\nu\gamma} \eta^{\mu\xi} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) + F^{\mu\nu} [\ \ ]$$

$$= \left(\partial^\gamma A^\xi - \partial^\xi A^\gamma \right) - \left(\partial^\xi A^\gamma - \partial^\gamma A^\xi \right) + F^{\gamma\xi} - F^{\xi\gamma}$$

$$= F^{\gamma\xi} - F^{\xi\gamma} + F^{\gamma\xi} - F^{\xi\gamma} = 4 F^{\gamma\xi}$$

Luego, las ecuaciones de E-L serán:

$$\partial_\gamma\left(\frac{\partial}{\partial(\partial_\gamma A_\xi)}\left[-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]\right)= \partial_\gamma\left(-F^{\gamma\xi}\right) = -\partial_\gamma F^{\gamma\xi} = 0 \rightarrow$$

$$\leftarrow \boxed{\partial_\gamma F^{\gamma\xi} = 0}$$

$$\partial_\gamma \partial^\gamma A^\xi - \partial_\gamma \partial^\xi A^\gamma = 0$$

$$\Box^2 A^\xi - \partial^\xi(\partial_\gamma A^\gamma) = 0$$

Consideramos entonces:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \text{con } \Lambda^{\mu\rho}\Lambda^{\nu\sigma}\eta_{\rho\sigma} = \eta^{\mu\nu}$$

a^μ constante

La corriente de Noether es:

$$J_\mu = -\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\frac{\partial\phi}{\partial x^\nu} - \eta_{\mu\nu}\mathcal{L}\right)\delta x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\cdot\delta\phi$$

y sometemos a una transformación de Lorentz

$$x'^\mu - x^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu - x^\mu \Rightarrow \text{consideraré separadamente} \begin{cases}\text{BOOST}\\ \text{ROTACIÓN}\end{cases}$$

, con la invariancia de Gauge

$$A'^\mu = A^\mu + \partial^\mu\xi \Rightarrow \delta A^\mu = \partial^\mu \xi$$

Para las coordenadas; separamos traslación y rotación:

$$x'^\mu = x^\mu + a^\mu \qquad\qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\downarrow \qquad\qquad \downarrow \text{ infinitesimal}$$

$$\delta x^\mu = a^\mu \qquad\qquad x'^\mu = (1 + \delta\omega^\mu{}_\nu)x^\nu \quad \text{con } \delta\omega^\mu{}_\nu = -\delta\omega_\nu{}^\mu$$

$$\downarrow \ \delta x^\mu = \delta\omega^{\mu\nu}x_\nu \rightarrow$$

- Para el lagrangiano L_I: considero $\delta x_\nu = \delta\omega_{\nu\beta}x^\beta$; $\delta A_\alpha = \delta\omega_{\alpha\Gamma}A^\Gamma$

$$J^\mu = -\Big[\underbrace{\frac{\partial\mathcal{L}}{\partial[\partial_\mu A_\alpha]}\cdot\partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L}}_{\equiv\,\Theta^{\mu\nu}}\Big]\delta x_\nu + \frac{\partial\mathcal{L}}{\partial[\partial_\mu A_\alpha]}\cdot\delta A_\alpha$$

$$J^\mu = -\Theta^{\mu\nu}\delta\omega_{\nu\beta}x^\beta + -F^{\mu\alpha}\delta\omega_{\alpha\Gamma}A^\Gamma$$

$$J^\mu = -\tfrac{1}{2}\Theta^{\mu\nu}\delta\omega_{\nu\beta}x^\beta - \tfrac{1}{2}\Theta^{\mu\nu}\delta\omega_{\nu\beta}x^\beta - F^{\mu\alpha}\delta\omega_{\alpha\Gamma}A^\Gamma$$

$$J^\mu = -\tfrac{1}{2}\Theta^{\mu\nu}\delta\omega_{\nu\beta}x^\beta - \tfrac{1}{2}\Theta^{\mu\beta}\delta\omega_{\beta\nu}x^\nu - F^{\mu\alpha}\delta\omega_{\alpha\Gamma}A^\Gamma;$$

$$J^\mu = -\tfrac{1}{2}\delta\omega_{\nu\beta}\left(\Theta^{\mu\nu}x^\beta - \Theta^{\mu\beta}x^\nu\right) - F^{\mu\alpha}\delta\omega_{\alpha\Gamma}A^\Gamma$$

$$J^\mu = -\frac{1}{2}\delta\omega_{\nu\beta}\left[\Theta^{\mu\nu}x^\beta - \Theta^{\mu\beta}x^\nu + F^{\mu\nu}A^\beta\right]$$

$$J^\mu = -\frac{1}{2}\delta\omega_{\nu\beta}M^{\mu\nu\beta}$$

Acá me cansé y la cuenta no la continúo. Sigo con ecuaciones de mov. para otros L, además el terreno parece menos seguro aquí.

- El lagrangiano $L_{II} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$ incorpora un término respecto de L_I $\Rightarrow$

$$\frac{\partial}{\partial[\partial_\nu A_\rho]}\left[-\tfrac{1}{2}(\partial_\mu A^\mu \partial^\alpha A_\alpha)\right] = -\tfrac{1}{2}\left[\frac{\partial(\partial_\mu A^\mu)}{\partial[\partial_\nu A_\rho]}\cdot\partial^\alpha A_\alpha + \partial_\mu A^\mu\cdot\frac{\partial(\partial^\alpha A_\alpha)}{\partial[\partial_\nu A_\rho]}\right]$$

$$= -\left(\partial_\mu A^\mu \cdot \frac{\partial(\partial^\alpha A_\alpha)}{\partial[\partial_\nu A_\rho]}\right)$$

$$= -\partial_\mu A^\mu \cdot \frac{\partial(\eta^{\alpha\beta}\partial_\beta A_\alpha)}{\partial[\partial_\nu A_\rho]} = -\partial_\mu A^\mu \cdot \eta^{\alpha\beta}\delta^\nu_\beta\delta^\rho_\alpha$$

$$= -\partial_\mu A^\mu \eta^{\rho\nu} = -\partial_\alpha A^\alpha \eta^{\mu\nu}$$ ← [Para estandarizar notación]

Entonces las ecuaciones de E-L resultan en:

$$\partial_\nu\left[-F^{\nu\mu} - \eta^{\mu\nu}\partial_\alpha A^\alpha\right] = 0$$

$$\partial_\nu F^{\nu\mu} + \partial^\mu(\partial_\alpha A^\alpha) = 0$$

$$\partial_\nu\partial^\nu A^\mu - \partial_\nu\partial^\mu A^\nu + \partial^\mu\partial_\nu A^\nu = 0$$

$$\partial_\nu\partial^\nu A^\mu - \partial^\mu \cancel{\partial_\nu} A^\nu + \partial^\mu\cancel{\partial_\nu} A^\nu = 0$$

$$\partial_\nu\partial^\nu A^\mu = 0 \Rightarrow \boxed{\Box^2 A^\mu = 0}$$

- El lagrangiano $L_{III} = -\frac{1}{2}A^\mu{}_{,\nu}A_\mu{}^{,\nu}$

$L_{III} = (-1/2)\,\partial_\nu A^\mu \partial^\nu A_\mu$

$$\frac{\partial\mathcal{L}}{\partial[\partial_\alpha A_\rho]} = (-1/2)\frac{\partial}{\partial[\partial_\alpha A_\beta]}\left(\partial_\nu \eta^{\mu\rho}A_\rho\, \eta^{\nu r}\partial_r A_\mu\right) = (-1/2)\,\eta^{\mu\rho}\eta^{\nu r}\left[\delta^\alpha_\nu\delta^\beta_\rho\cdot\partial_r A_\mu + \partial_\nu A_\rho\,\delta^\alpha_r\delta^\beta_\mu\right]$$

$$= (-1/2)\left[\eta^{\mu\beta}\eta^{\alpha r}\partial_r A_\mu + \eta^{\beta\rho}\eta^{\nu\alpha}\partial_\nu A_\rho\right] = (-1/2)\left[\partial^\alpha A^\beta + \partial^\alpha A^\beta\right]$$

$$= -\partial^\alpha A^\beta \Rightarrow$$

Las ecuaciones de E-L son:

$$\partial_\alpha\left[-\partial^\alpha A^\beta\right] = -\partial_\alpha\partial^\alpha A^\beta = 0 \rightarrow \boxed{\Box^2 A^\beta = 0}$$

- El Lagrangiano: $L_{IV} = \frac{1}{2}\left[A_\mu(\partial_\nu F^{\mu\nu}) - (\partial_\mu A_\nu)F^{\mu\nu}\right] + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

, considera a A_μ, $F^{\mu\nu}$ como variables independientes. Para A_β serán:

$$\frac{\partial\mathcal{L}}{\partial[\partial_\alpha A_\beta]} = \frac{\partial}{\partial[\partial_\alpha A_\beta]}\left(\tfrac{1}{2}[-\partial_\mu A_\nu]F^{\mu\nu}\right) = (-1/2)\,\delta^\alpha_\mu\delta^\beta_\nu F^{\mu\nu} = -1/2\, F^{\alpha\beta}$$

$$\frac{\partial\mathcal{L}}{\partial A_\beta} = 1/2\,\delta^\beta_\mu(\partial_\nu F^{\mu\nu}) = 1/2\,\partial_\nu F^{\beta\nu}$$

Entonces, para el campo A_β las ecuaciones de E-L son:

$$-1/2\ \partial_\alpha F^{\alpha\beta} - 1/2\ \partial_\nu F^{\beta\nu} = 0$$

$$-1/2\left(\partial_\alpha F^{\alpha\beta} - \partial_\nu F^{\nu\beta}\right) = 0 \quad \rightarrow \quad 0 = 0 \qquad \text{No me da información}$$

Para el campo $F^{\mu\nu}$ será:

$$\frac{\partial \mathcal{L}}{\partial[\partial_\alpha F^{\beta\gamma}]} = 1/2\ A_\mu\, \delta^\alpha_\nu \delta^\mu_\beta \delta^\nu_\gamma = 1/2\ A_\beta\, \delta^\alpha_\nu \delta^\nu_\gamma = \frac{1}{2} A_\beta\, \delta^\alpha_\gamma$$

$$\frac{\partial \mathcal{L}}{\partial F^{\beta\gamma}} = -1/2\left[\partial_\mu A_\nu . \delta^\mu_\beta \delta^\nu_\gamma\right] + 1/4\left(\delta^\mu_\beta \delta^\nu_\gamma . F_{\mu\nu} + F^{\mu\nu} \eta_{\mu\rho} \eta_{\nu\sigma}\, \delta^\rho_\beta \delta^\sigma_\gamma\right)$$

$$= -1/2\left(\partial_\beta A_\gamma\right) + \frac{1}{4}\left(F_{\beta\gamma} + F^{\mu\nu} \eta_{\mu\beta} \eta_{\nu\gamma}\right)$$

$$= -\frac{1}{2}\ \partial_\beta A_\gamma + \frac{1}{4}\left(F_{\beta\gamma} + F_{\beta\gamma}\right) = -\frac{1}{2}\ \partial_\beta A_\gamma + \frac{1}{2} F_{\beta\gamma}$$

Luego, las ecuaciones de E-L serán:

$$1/2\ \partial_\alpha A_\beta \delta^\alpha_\gamma + 1/2\ \partial_\beta A_\gamma - 1/2\ F_{\beta\gamma} = 0$$

$$\frac{1}{2}\left(\partial_\gamma A_\beta + \partial_\beta A_\gamma - F_{\beta\gamma}\right) = 0$$

$$\boxed{\partial_\gamma A_\beta + \partial_\beta A_\gamma - F_{\beta\gamma} = 0}$$

7.

$$\mathcal{L} = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi \phi^*$$

(a) transformación $\phi \to \phi' = e^{i\alpha(x^\mu)} \phi$

$$\phi'^* = e^{-i\alpha(x^\mu)} \phi^*$$

$$\partial_\mu \phi' = \partial_\mu \left(e^{i\alpha(x)} \phi \right) = e^{i\alpha(x)}\, i\, \partial_\mu \alpha(x) \cdot \phi + e^{i\alpha(x)} \partial_\mu \phi$$

$$\partial^\mu \phi'^* = \partial^\mu \left(e^{-i\alpha(x)} \phi^* \right) = -e^{-i\alpha(x)}\, i\, \partial^\mu \alpha(x) \cdot \phi^* + e^{-i\alpha(x)} \partial^\mu \phi^*$$

Según se ve:

$$\partial_\mu \phi\, \partial^\mu \phi^* \neq \partial_\mu \phi'\, \partial^\mu \phi^{*\prime}$$

$$\neq \left(e^{i\alpha} i \partial_\mu \alpha \cdot \phi + e^{i\alpha(x)} \partial_\mu \phi \right) \cdot \left(-e^{-i\alpha} i \partial^\mu \alpha \cdot \phi^* + e^{-i\alpha} \partial^\mu \phi^* \right)$$

$$\partial_\mu \phi\, \partial^\mu \phi^* \neq \partial_\mu \alpha \cdot \phi \cdot \partial^\mu \alpha \cdot \phi^* + i \partial_\mu \alpha \cdot \phi\, \partial^\mu \phi^* - i \partial_\mu \phi \cdot \partial^\mu \alpha \cdot \phi^* + \partial_\mu \phi\, \partial^\mu \phi^*$$

$\Rightarrow$ $\boxed{\mathcal{L} \text{ No es invariante frente a } \phi \to \phi'}$

(b) La idea es pedir una derivada nueva tal que:

$$D_\mu \phi \longrightarrow \begin{cases} (D_\mu \phi)' = e^{i\alpha(x)} D_\mu \phi \\ (D^\mu \phi^*)' = e^{-i\alpha(x)} D^\mu \phi^* \end{cases}$$

$\Rightarrow$ se propone $D_\mu = \partial_\mu + ieA_\mu$

Pretendemos $(D_\mu \phi)' = e^{i\alpha} D_\mu \phi$ [1]

$$D'_\mu \phi' = e^{i\alpha} D_\mu \phi \Rightarrow$$

$$\partial_\mu \phi' + ieA'_\mu \phi' = \partial_\mu \phi \cdot e^{i\alpha} + e^{i\alpha} i \partial_\mu \alpha \cdot \phi + ie\, e^{i\alpha} A'_\mu \phi$$

$$= e^{i\alpha} \left[\partial_\mu \phi + i \partial_\mu \alpha \cdot \phi + ie A'_\mu \phi \right]$$

$$= e^{i\alpha} \left[\partial_\mu + ie \left(\frac{1}{e} \partial_\mu \alpha + A'_\mu \right) \right] \phi$$

ahora usando [1] $\to$

$$= e^{i\alpha} \left[\partial_\mu + ieA_\mu \right] \phi$$

$\Rightarrow$ $A_\mu = \frac{1}{e} \partial_\mu \alpha + A'_\mu$ $\therefore$ $\boxed{A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha(x)}$

Esta regla de transformación se parece a la del cuadripotencial electromagnético. Es la transformación de Gauge del EM.

(c)
$$\mathcal{L} = D_\mu \phi (D^\mu \phi)^* - m^2 \phi \phi^*$$

$$\phi \to \phi' = e^{i\alpha(x)} \phi$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

$$\phi' \phi'^* = e^{i\alpha} \phi \cdot e^{-i\alpha} \phi^* = \phi \phi^*$$

$$D'_\mu \phi' = \left(\partial_\mu + ie A_\mu - i \partial_\mu \alpha(x)\right) e^{i\alpha} \phi$$

$$e^{i\alpha} i \partial_\mu \alpha \phi + e^{i\alpha} \partial_\mu \phi + ie e^{i\alpha} A_\mu \phi - i \partial_\mu \alpha e^{i\alpha} \phi$$

$$= e^{i\alpha} [\partial_\mu + ie A_\mu] \phi = e^{i\alpha} D_\mu \phi$$

$$D'^\mu \phi^{*\prime} = \left(\partial^\mu - ie A^\mu + i \partial^\mu \alpha\right) e^{-i\alpha} \phi^*$$

$$-e^{-i\alpha} i \partial^\mu \alpha \cdot \phi^* + e^{-i\alpha} \partial^\mu \phi^* - ie e^{-i\alpha} A^\mu \phi^* + i \partial^\mu \alpha \cdot e^{-i\alpha} \phi^*$$

$$= e^{-i\alpha} [\partial^\mu - ie A^\mu] \phi^* = e^{-i\alpha} D^{\mu *} \phi^*$$

Luego
$$(D_\mu \phi)' (D^\mu \phi)^{*\prime} = e^{i\alpha} D_\mu \phi \cdot e^{-i\alpha} (D^\mu \phi)^* = D_\mu \phi (D^\mu \phi)^*$$

$$\Rightarrow \quad \mathcal{L} = (D_\mu \phi)' (D^\mu \phi)^{*\prime} - m^2 \phi' \phi^{*\prime} = D_\mu \phi (D^\mu \phi)^* - m^2 \phi \phi^*$$

$\therefore$ $\boxed{\mathcal{L} \text{ es invariante ante la transformación de fase global}}$

(d)
$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^*$$

$$\mathcal{L} = (\partial_\mu \phi + ie A_\mu \phi)(\partial^\mu \phi^* - ie A^\mu \phi^*) - m^2 \phi \phi^*$$

$$\mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi^* + ie A_\mu \phi (\partial^\mu \phi^*) - ie (\partial_\mu \phi) A^\mu \phi^* + e^2 A_\mu \phi \cdot A^\mu \phi^*$$
$$- m^2 \phi \phi^*$$

Como lo que varía son los campos $\Rightarrow$

$$J_\rho = \sum_i \frac{\partial \mathcal{L}}{\partial [\partial^\rho \phi_i]} \delta \phi_i \quad \therefore$$

- $\delta \phi = i\alpha(x) \phi$
- $\delta \phi^* = -i\alpha(x) \phi^*$
- $\delta A_\mu = A'_\mu - A_\mu = \cancel{A_\mu} - \frac{1}{e} \partial_\mu \alpha(x) - \cancel{A_\mu}$

$$\delta A_\mu = -\frac{1}{e} \partial_\mu \alpha(x)$$

$$\frac{\partial \mathcal{L}}{\partial [\partial^\rho \phi]} = \frac{\partial [\partial_\mu \phi \cdot \partial^\mu \phi^* - ie A^\mu \phi^* (\partial_\mu \phi)]}{\partial [\partial^\rho \phi]}$$

$$= \frac{\partial [\eta_{\mu\alpha} \partial^\alpha \phi \cdot \partial^\mu \phi^* - ie A^\mu \phi^* \eta_{\mu\alpha} \partial^\alpha \phi]}{\partial [\partial^\rho \phi]} = [\eta_{\mu\alpha} \delta^\alpha_\rho \partial^\mu \phi^* - ie A^\mu \phi^* \eta_{\mu\alpha} \delta^\alpha_\rho]$$

$$= \eta_{\mu\rho} \partial^\mu \phi^* - ie A^\mu \phi^* \eta_{\mu\rho} = \partial_\rho \phi^* - ie A_\rho \phi^* = (D_\rho \phi)^*$$

$$\frac{\partial \mathcal{L}}{\partial[\partial^\rho \phi^*]} = \frac{\partial}{\partial[\partial^\rho\phi^*]}\left[\partial_\mu\phi\cdot\partial^\mu\phi^* + ieA_\mu\phi\,\partial^\mu\phi^*\right]$$

$$= \partial_\rho\phi + ieA_\mu\phi\,\delta^\mu_\rho = \partial_\rho\phi + ieA_\rho\phi = D_\rho\phi$$

$$\frac{\partial \mathcal{L}}{\partial[\partial^\rho A_\rho]} = \frac{\partial}{\partial[\partial^\rho A_\rho]}[\,0\,] = 0$$

$$J'_\rho = (D_\rho\phi)^*\cdot i\alpha(x)\phi - (D_\rho\phi)\cdot i\alpha(x)\phi^*$$

$$J'_\rho = i\alpha(x)\,[(D^*_\rho\phi^*)\phi - (D_\rho\phi)\phi^*]$$

$$J'_\rho = i\alpha(x)\left[\partial_\rho\phi^*\cdot\phi - ieA_\rho\phi^*\phi - \partial_\rho\phi\cdot\phi^* - ieA_\rho\phi\cdot\phi^*\right]$$

$$J'_\rho = i\alpha(x)\left[\phi\cdot\partial_\rho\phi^* - \phi^*\partial_\rho\phi - ieA_\rho 2\phi^*\phi\right]$$

La cantidad conservada tiene metida a la corriente EM. Escribiendo el $\mathcal{L}$ y agrupando los términos en A_μ obtenemos:

$A^\mu = \eta^{\mu\rho}A_\rho$

$$\mathcal{L} = \partial_\mu\phi\cdot\partial^\mu\phi^* + ieA_\mu[\phi\cdot\partial^\mu\phi^* - \underbrace{\partial_\mu\phi\cdot\eta^{\mu\rho}\phi^*}_{\partial^\mu\phi\cdot\phi^*} \underbrace{- ieA^\mu\phi\phi^*]}_{\text{término de interacción } A^\mu \text{ con } \phi,\phi^*} \underbrace{- m^2\phi\phi^*}_{\text{asociado a la masa del } \phi}$$

lo de adentro tiene muchas ganas de ser la corriente, aunque no llega a serlo

(e) Si $\phi\in\mathbb{R}$ → $\phi^* = \phi$ → Se pierde la invariancia frente a $\phi\to\phi' = e^{i\alpha(x)}\phi$ pues

$$m^2\phi'^2 = m^2 e^{i2\alpha(x)}\phi^2 \neq m^2\phi^2$$

Entonces al pedir invariancia ante $U(1)$ local vemos que surge el EM. Faltaría meter un término cinético para el A_μ, cosa de que pueda propagarse este campo, y tenemos un $\mathcal{L}$ de QED escalar:

$$\mathcal{L} = D_\mu\phi\,(D^\mu\phi)^* - m^2\phi\phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Un campo escalar real $\phi=\phi^*$ no es invariante frente a $U(1)$ local, con lo cual no interactúa con el EM. Diremos en este caso que es "neutro" en oposición al caso complejo que sí interactúa y por ello llamaremos "cargado".

8.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi \phi + \lambda \phi^n \qquad n>2$$

tenemos un solo campo $\phi \rightarrow$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi + \lambda n \phi^{n-1}$$

$$\frac{\partial \mathcal{L}}{\partial[\partial_\alpha \phi]} = \frac{\partial}{\partial[\partial_\alpha \phi]}\left(1/2\, \partial_\mu \phi\, \eta^{\mu\rho} \partial_\rho \phi \right) = 1/2\, \eta^{\mu\rho}\left(\delta_\mu^\alpha \partial_\rho \phi + \partial_\mu \phi\, \delta_\rho^\alpha \right)$$

$$= 1/2 \left(\eta^{\alpha\rho} \partial_\rho \phi + \eta^{\mu\alpha} \partial_\mu \phi \right)$$

$$= 1/2 \left(\partial^\alpha \phi + \partial^\alpha \phi \right) = \partial^\alpha \phi$$

La ecuaciones de E-L serán

$$\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial[\partial_\alpha \phi]} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\partial_\alpha \left(\partial^\alpha \phi \right) + m^2 \phi - \lambda n \phi^{n-1} = 0$$

$$\boxed{\partial_\alpha \partial^\alpha \phi + m^2 \phi - \lambda n \phi^{n-1} = 0}$$

Con $\lambda = 0$ es:

$$\partial_\alpha \partial^\alpha \phi + m^2 \phi = 0$$

ecuación de Klein-Gordon para un campo escalar Neutro, con m^2 la constante asociada a la masa.

con $\lambda \neq 0$ es

$$\partial_\alpha \partial^\alpha \phi + m^2 \phi + (-\lambda n)\, \phi^{n-1} = 0$$

Si $n = 2$ tenemos

$$\partial_\alpha \partial^\alpha \phi + (m^2 - \lambda 2)\, \phi = 0$$

$\rightarrow$ Ecuación de Klein-Gordon con un término de masa modificada

Si $n = 3$ tenemos

$$\partial_\alpha \partial^\alpha \phi + m^2 \phi - 3\lambda \phi \phi = 0$$

$\rightarrow$ La ecuación ha dejado de ser lineal $\rightarrow$ la Σ soluciones no es solución

Como consecuencia de esto último (no linealidad de ϕ) surge la interacción entre las partículas representadas por el campo $\phi(\vec{x}, t)$.

Luego, términos cúbicos en la $\mathcal{L}$ (que devienen en términos cuadráticos en las ecuaciones de E-L) corresponden a interacción entre los campos.